MAGNETOSTATICS

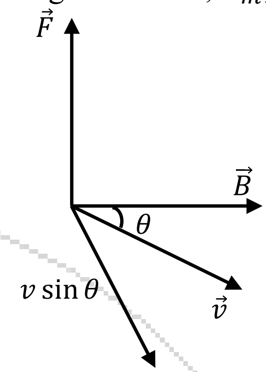
BSc – I SEM – II (UNIT IV)

Contents

Magnetic Induction Field Vector ($\vec{B}$)

If a test charge, q, moves with a velocity, $\vec{v}$, through a point P and experiences a force $\vec{F}_m$ in addition to the electric field, which changes with magnitude as well as direction of velocity, $\vec{v}$, then magnetic induction field $\vec{B}$ is said to exist at P.

If we vary the direction of $\vec{v}$ through the point P, keeping magnitude of $\vec{v}$ constant, then it is observed that the magnitude of the force $\vec{F}_m$ changes but its direction always remains perpendicular to the direction of $\vec{v}$. For a particular direction of $\vec{v}$, the magnetic force, $\vec{F}_m$, is zero. This direction gives the direction of the magnetic field $\vec{B}$. When $\vec{v}$ is perpendicular to this direction, the magnitude of the magnetic force, F_m, is maximum.



$$|\vec{B}| = \frac{|(\vec{F}_m)_{max}|}{qv}$$

$$\vec{F}_m = q(\vec{v}\times\vec{B})$$

$$= qvB\sin\theta$$

where, θ is the angle that direction of $\vec{v}$ makes with the direction of $\vec{B}$.

The SI unit of magnetism is Tesla (T) and the CGS unit is Gauss (G). ($1T = 10^4 G$)

Lorentz Force Law

The force on a test charge, q, when it moves with a velocity, $\vec{v}$, through a region in which both electric field, $\vec{E}$, and magnetic field, $\vec{B}$, are present is given by,

$$\vec{F} = q\vec{E} + q(\vec{v}\times\vec{B})$$

This relation is known as the Lorentz force law and the force $\vec{F}$ is called as the Lorentz force.

Motion of Charged Particle in a Uniform Magnetic Field

A static magnetic field does not exert any force on a charged particle at rest. It can experience a magnetic force only when it enters the magnetic field with some velocity $\vec{v}$. The force is given as

$$\vec{F}_m = q(\vec{v}\times\vec{B}) = qvBsin\theta$$

The direction of the force is along the normal to the plane containing the vectors $\vec{v}$ and $\vec{B}$. This means that the Lorentz force is always perpendicular to the displacement of the charged particle.

If $d\vec{x}$ is the infinitesimal displacement of charged particle during the time interval dt, the work done dW on charged particle by the magnetic field is given by

$$dW = \vec{F}_m . d\vec{x} = \vec{F}_m . \vec{v}dt = q(\vec{v}\times\vec{B}).\vec{v}dt = 0$$

It implies that a steady magnetic field does no work when the charged particle is displaced. This means that the kinetic energy of the charged particle does not change due to the action of the magnetic field.

As the force $\vec{F}_m$ and velocity $\vec{v}$ are perpendicular to each other, we get

$$\vec{F}_m . \vec{v} = mav\, cos90^0 = 0$$

$$= m\frac{dv}{dt}v = \frac{d}{dt}\left(\frac{v^2}{2}\right) = 0$$

$$\frac{1}{2}mv^2 = constant$$

$$v = constant$$

The applied magnetic field can only change the direction of the velocity vector but can’t alter the speed of the moving charged particle.

i) **Motion parallel to the Magnetic Field (Longitudinal Field):**

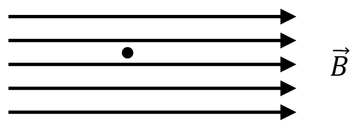
If a charged particle moves along the magnetic lines of induction, the Lorentz force is given by

$$F_m = qvB sin\theta = qvB \sin 0 = 0$$

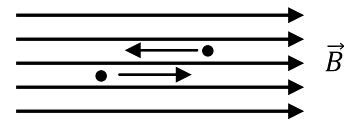
Similarly, if the charged particle moves opposite to the field lines then the Lorentz force is given by

$$F_m = qvB sin\theta = qvB \; sin\,(\pi/2) = 0$$

In both the cases the charged particle continues to move along the initial direction of motion without a change in speed or direction.

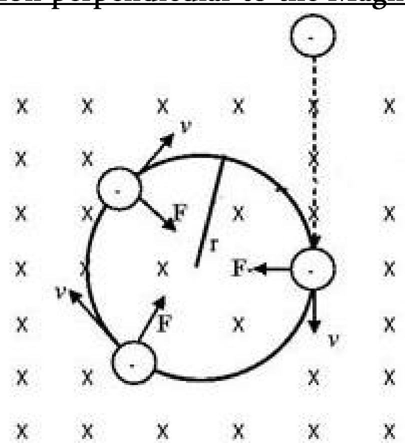


$v = 0, \; F_m = 0$

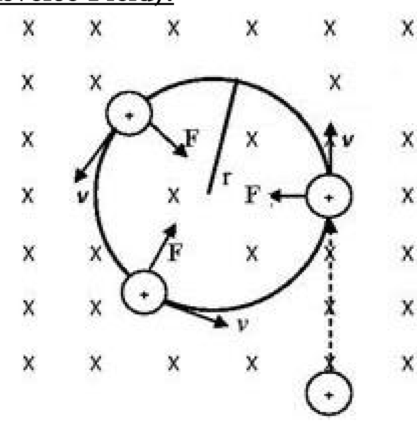


$v \parallel B, \; F_m = 0$

ii) **Motion perpendicular to the Magnetic Field (Transverse Field):**



Negatively charged particle



Positively charged particle

Let us now consider the case of an electron entering a uniform magnetic field $\vec{B}$ with its initial velocity vector $\vec{v}$ normal to the field. The magnetic field is considered to be having a direction into the plane of the paper, indicated by the crosses. The magnetic force is given by,

$$F_m = qvB$$

which has a constant magnitude. This force can't change the magnitude of electron velocity but can deflect the electron continuously along a curvilinear path. The tangential component of $\vec{F}_m$ will be zero as $\vec{F}_m$ and velocity $\vec{v}$ are mutually perpendicular. Therefore, the normal component will be equal to $\vec{F}_m$ itself. This will always act perpendicular to $\vec{v}$ at each point. Therefore, $\vec{F}_m$ is a centripetal force. According to Newton's laws of motion, centripetal force is given by

$$F_C = \frac{mv^2}{r}$$

Therefore,

$$qvB = \frac{mv^2}{r}$$

or,

$$r = \frac{mv}{qB}$$

Since all parameters are constant, $r = constant$. Therefore, the path followed by the charged particle would be a circle. Thus, the charged particle will describe a circular path in a plane perpendicular to the magnetic induction lines. The sense of rotation will be clockwise if the magnetic field is directed into the plane of paper. For a positive charge the sense of rotation will be anti-clockwise.

The time period of one revolution is given by

$$T = (Distance\ covered\ in\ one\ revolution) / (Speed\ of\ the\ particle)$$

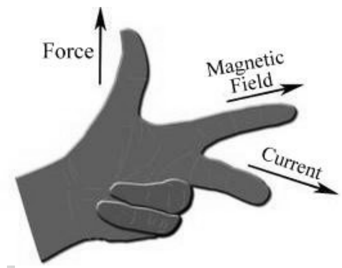
$$T = \frac{2\pi r}{v} = \left(\frac{2\pi}{v}\right)\left(\frac{mv}{qB}\right) = \frac{2\pi m}{qB}$$

The frequency of revolution of the object is given by

$$f = \frac{1}{T} = \frac{qB}{2\pi m}$$

This indicates that the time period and frequency are independent of the velocity of the charged particle and radius of the circular path. v and r adjust such that T and f stay constant.
Thus, slower particles move in smaller circles while faster particles move in larger circles so as to keep the time period constant.

Fleming's Left Hand Rule



If we stretch the index finger, the middle finger and the thumb of left hand mutually perpendicular to each other such that the index finger points in the direction of magnetic field and the middle finger points in the direction of the motion of positive charge (i.e. current) then the thumb points in the direction of the force experienced by the charged particle.

Force on a Conductor Carrying Current In Uniform Magnetic Field

Consider a wire of length $\vec{l}$ and cross – sectional area $\vec{A}$ in which current I is flowing from left to right. Let us suppose that that the positive charge carriers are moving with velocity $\vec{v}$ in the direction of current. Let N be the number of charge carriers per unit volume. Let us consider a current element $\overrightarrow{dl}$. The volume of this current element will be, $dV = A\ dl$. Hence, the number of charge carriers in the volume dV will be

$$N\ dV = N\ Adl.$$

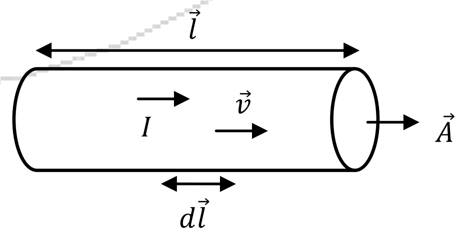
When magnetic field, $\vec{B}$, is applied, the Lorentz force on the current element is given by,

$$d\vec{F} = (N\ Adl\ q)(\vec{v} \times \vec{B})$$

The current in a wire is given by,

$$I = NqvA$$

Hence, the Lorentz force is given by,



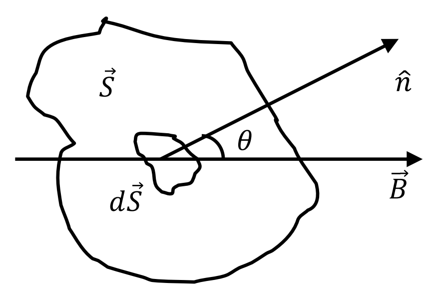
$$d\vec{F} = dl[(N\ A\ q)\vec{v} \times \vec{B}]$$
$$= dl(I \times \vec{B})$$
$$= I(d\vec{l} \times \vec{B})$$

Integrating over the entire length of the wire, we get

$$\vec{F} = \int d\vec{F} = I(\vec{l} \times \vec{B})$$

$$F = IlB \sin\theta$$

Magnetic Flux



Magnetic flux through a surface is defined as the total number of lines of magnetic induction passing through the surface.
Let us consider an element of surface area $d\vec{S}$ of a surface area $\vec{S}$. Let $\hat{n}$ be the unit vector perpendicular to the area element $d\vec{S}$, then $\hat{n}$ represents the direction of the area vector $d\vec{S}$. The net outward flow or flux $d\phi$ is the average outdrawn normal component of the magnetic field $\vec{B}$ times the area of the surface.

$$d\phi = (B\cos\theta).dS$$
$$= \vec{B}.\hat{n}\,dS$$

Hence, the total flux is given by,

$$\phi = \int d\phi = \int_S \vec{B}.\hat{n}\,dS = \int_S (B\cos\theta).dS$$

If $\vec{B}$ is uniform over the area $\vec{S}$, then

$$\phi = (B\cos\theta)\int_S dS = BS\cos\theta$$

For, $\theta = 0$, $B\cos\theta = B$. Hence, we get

$$d\phi = BdS$$
$$B = \frac{d\phi}{dS}$$

Magnetic field, B, is thus defined as the magnetic flux unit per area. The SI unit for magnetic flux is Weber.

Gauss's Law in Magnetostatics

The field lines of a static electric field originate and end on electric charges. On the other hand, magnetic lines of force are continuous, i.e. they have no sources or sinks. When such kinds of curves are obtained, the field is called solenoidal. Hence, magnetic field is a solenoidal field. Since magnetic field, $\vec{B}$, is continuous, the magnetic field entering any region is equal to the flux leaving it. Hence, over a closed surface,

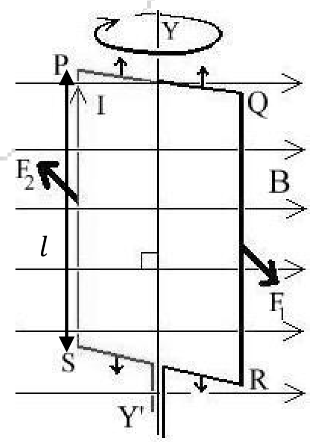
$$\int_S \vec{B}.d\vec{S} = 0$$

This is the Gauss's law applied to the magnetic field. Since, the flux entering any volume is equal to the flux leaving, the net flux over the volume is zero. Hence, at any point,

$$div\,\vec{B} = \vec{\nabla}.\vec{B} = 0$$

This is one of the Maxwell's laws of electrodynamics. This proves that magnetic monopoles do not exist.

Rectangular Current Loop in External Magnetic Field

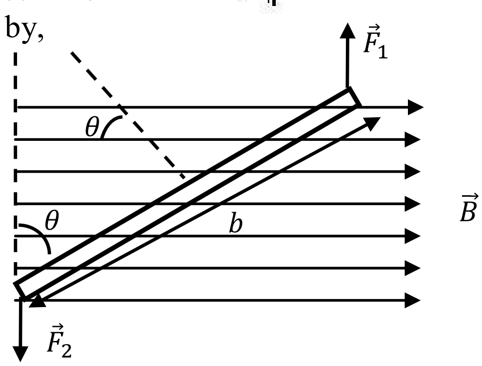


Let us consider a rectangular loop $PQRS$ carrying current, I, in the clockwise direction placed in a uniform magnetic field of induction $\vec{B}$. Let $\vec{F_1}$ be the force acting on side QR. As $\vec{l}$ is perpendicular to $\vec{B}$, the force $\vec{F_1}$ is perpendicular to $I\vec{l}$ and $\vec{B}$. Thus,

$$\vec{F_1} = I(\vec{l}\times\vec{B}) = IlB$$

Similarly, there is a force $\vec{F_2}$ acting on the side SP which is equal and opposite to $\vec{F_1}$. The sides PQ and RS experience a net force which is zero.

The two forces $\vec{F_1}$ and $\vec{F_2}$ being equal, opposite and parallel constitute a couple about an axis YY'. The perpendicular distance called as moment arm between the two forces is $bsin\theta$. Hence, the moment of the couple i.e. the torque is given by,



$$Torque = Force \times Moment\ Arm$$
$$\tau = F_1(bsin\theta)$$
$$= (IlB)(bsin\theta)$$
$$= IBAsin\theta$$

where, $A = lb$ is the area of the loop. The above expression can be expressed in vector form as,

$$\vec{\tau} = I(\vec{A}\times\vec{B})$$

For n turns per unit length, the above expression is,

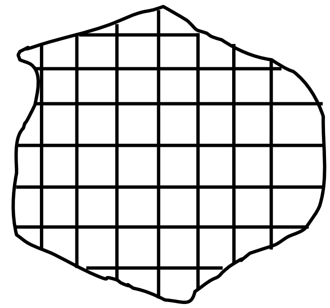
$$\vec{\tau} = (nI\vec{A}\times\vec{B})$$

The quantity $nI\vec{A}$ is called as the magnetic dipole moment of the loop, $\vec{m}$. Hence,

$$\vec{\tau} = \vec{m} \times \vec{B}$$

The direction of the torque is perpendicular to the plane containing the vectors $\vec{m}$ and $\vec{B}$. This expression is valid for a loop of any arbitrary shape. The SI unit of magnetic dipole moment is $Amp - metre$.

Equivalence of a Current Coil to a Bar Magnet



Let us consider a coil of arbitrary shape containing current I and let its total area be A. Let us divide the plane of the coil into large number of narrow rectangular circuits. The current in the coil remains the same as in the common boundary of the two adjacent loops; the two are having equal but oppositely directed currents. Hence, they mutually cancel each other. The torque due to i[th] rectangular circuit is

$$\tau_i = IA_iB\sin\theta$$

The total torque is given by,

$$\tau = \sum_{i=1}^{n} \tau_i = IB\sin\theta \sum_{i=1}^{n} A_i = IAB\sin\theta = mB\sin\theta$$

which is the expression for the torque on any bar magnet kept in a magnetic field. Hence, any current carrying coil is equivalent to a bar magnet or a magnetic dipole.

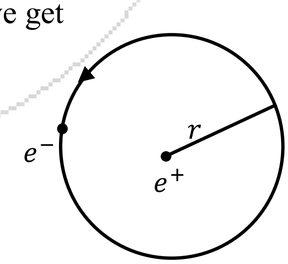
Magnetic Moment, Angular Momentum and Gyromagnetic Ratio of an Atomic Dipole

All materials are composed of atoms and an atom consists of a central positively charged nucleus. A specific number of electrons revolve around this nucleus in more or less circular orbits. The revolution of an electron in a clockwise direction is equivalent to a conventional current in the anticlockwise direction and the electronic orbit behaves like a magnetic dipole.

Let us consider Hydrogen. It consists of an electron of charge, $e = -1.6 \times 10^{-19} C$, revolving around a proton in the nucleus. Let the mass of the electron be m and its velocity of revolution be v about the circular orbit of radius r. The electron remains in the circular orbit due to the electrostatic force between the electron and the nucleus which provides the required centripetal force. Taking magnitudes, we get

$$Centripetal\ Force = Electrostatic\ Force$$

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$



$$v = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{mr}}$$

Now, $m = 9.1 \times 10^{-31} kg$, $e = 1.6 \times 10^{-19} C$, $r = 0.5Å$ and $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9\ Nm^2/C^2$. Hence, we get

$$v = 2.262 \times 10^6\ m/s$$

And the time period of one revolution is,

$$T = \frac{2\pi r}{v} = \frac{2\pi \times 0.5 \times 10^{-10}}{2.262 \times 10^6} = 1.39 \times 10^{-16} seconds$$

We find that, the time taken by the electron for one revolution is very very small and hence the electron will go round the nucleus million of times during the time of observation. Hence, the electron does not behave as an isolated particle but will appear as a current loop carrying current I, given by

$$I = \frac{Charge}{Time\ period\ of\ 1\ revolution} = \frac{e}{2\pi r/v} = \frac{ev}{2\pi r}$$

The magnitude of the dipole magnetic moment of the current loop is given by,

$$m_A = current\ \times area\ of\ the\ loop$$
$$m_A = IA = \frac{ev}{2\pi r}\pi r^2$$
$$m_A = \frac{1}{2}evr = \frac{e}{2m}(mvr) = \frac{e}{2m}L$$

where, $L = mvr$ is the angular momentum of the revolving electron. This is called the orbital magnetic moment.

According to Bohr's theory, an electron can only revolve in designated orbits around the nucleus in which the total angular momentum is an integral multiple of $\frac{h}{2\pi}$, where h is the Planck's constant. Therefore,

$$L = n\frac{h}{2\pi}$$

Hence,

$$m_A = \frac{e}{2m}\frac{nh}{2\pi} = \frac{ne}{2m}\hbar$$

where, $\hbar = h/2\pi$. For Hydrogen, $n = 1$. Thus, we get

$$m_A = \frac{eh}{4\pi m}$$

This is the unit of magnetic moment and is known as Bohr Magneton. It is denoted by μ_B.

$$\mu_B = \frac{eh}{4\pi m}$$

Putting the value of $e = 1.6 \times 10^{-19}C$, $m = 9.1 \times 10^{-31}kg$ and $h = 6.6 \times 10^{-34}\ Jsec$, we get

$$\mu_B = 9.27 \times 10^{-24}J/T$$

The ratio of the magnetic moment of the atomic dipole to its angular momentum is defined as the orbital gyromagnetic ratio. Thus,

$$Orbital\ gyromagnetic\ ratio = \frac{m_A}{L} = \frac{e}{2m}$$

This ratio is also known as the magneto – mechanical ratio.

Biot Savart's Law

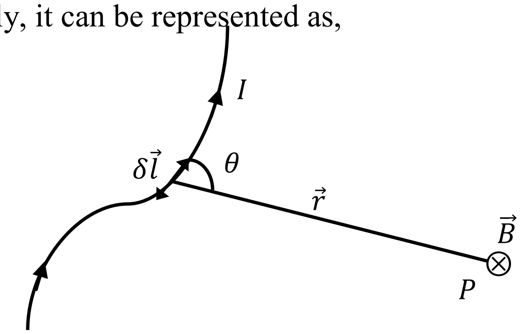
Let us consider a length δl of a conductor carrying current I. The magnitude δB of the magnetic induction at a point P due to the current element $I\delta l$ is given by,

$$\delta B \propto \frac{I\delta l\sin\theta}{r^2}$$
$$\delta B = \frac{\mu_0}{4\pi}\frac{I\delta l\sin\theta}{r^2}$$

This is the Biot Savart's Law.

Hence, it is observed that the magnitude is directly proportional to the current element $I\delta l$ and the sine of angle θ between the current element $I\delta\vec{l}$ and vector $\vec{r}$. The magnetic field is also inversely proportional to the distance of point P from the current element $I\delta l$. Vectorically, it can be represented as,

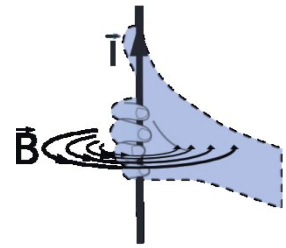
$$\delta\vec{B} = \frac{\mu_0}{4\pi}\frac{I(\delta\vec{l} \times \hat{r})}{r^2}$$
$$\delta\vec{B} = \frac{\mu_0}{4\pi}\frac{I(\delta\vec{l} \times \vec{r})}{r^3}$$



Hence, total magnetic field is given by,

$$\vec{B} = \oint d\vec{B} = \frac{\mu_0}{4\pi}\oint\frac{I(d\vec{l} \times \vec{r})}{r^3}$$

For surface and volume currents, the Biot Savart's law can be expressed respectively as,



$$\vec{B} = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}\times\vec{r}}{r^3}dA$$
$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\vec{r}}{r^3}d\tau$$

where, dA and $d\tau$ are area and volume elements while $\vec{K}(r)$ and $\vec{J}(r)$ are functions of r.

If we grasp the wire with our right hand with the thumb pointing towards the direction of current, then the fingers will curl around the wire in the direction of magnetic field. This is known as the Right hand thumb rule.

Applications

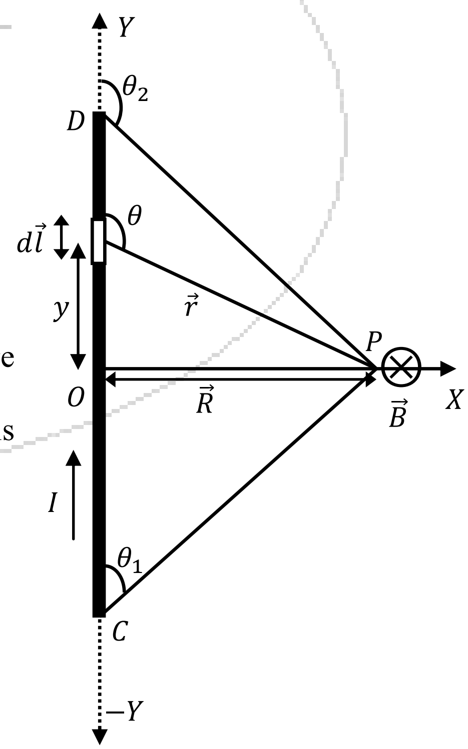
i) **Magnetic Field due to a Long Straight Conductor Carrying a Steady Current**

Let CD be a straight conductor carrying current I. The conductor is taken along the Y axis and the current is supposed to flow from C to D. Consider a point P at a perpendicular distance $R = OP$ from the conductor where O represents the origin and OP is on the X axis.

Let $d\vec{l}$ be a small current element at a distance y from the origin O and let the vector distance of P from the centre of the element be $\vec{r}$. Hence, according to laws of vector addition, we get

$$\vec{r} = -y\hat{j} + R\hat{\imath}$$

Hence, $d\vec{l} = dy\hat{j}$



The magnitude field according to Biot Savart's law is given as,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I(d\vec{l}\times\vec{r})}{r^3}$$
$$= \frac{\mu_0 I}{4\pi}\frac{[dy\hat{j}\times(-y\hat{j}+R\hat{\imath})]}{r^3}$$
$$= \frac{\mu_0 I}{4\pi}\frac{Rdy}{r^3}(-\hat{k})$$

Thus, the direction of $d\vec{B}$ at point P is into the plane of the figure at right angle to the page.

The magnitude of the magnetic field due to the entire wire is given as,

$$B = \int_C^D dB = \int_C^D \frac{\mu_0 I}{4\pi}\frac{Rdy}{r^3}$$

From the figure, we obtain

$$R = r\sin(\pi-\theta) = r\sin\theta \ \Rightarrow r = R\,cosec\,\theta$$
$$y = R\cot(\pi-\theta) = -R\cot\theta$$
$$dy = R\,cosec^2\theta\, d\theta$$

At C, $\theta = \theta_1$ and at D, $\theta = \theta_2$. Hence, we get

$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{(r\sin\theta)(R\,cosec^2\theta\, d\theta)}{(R\,cosec\,\theta)^3}$$
$$= \frac{\mu_0 I}{4\pi R}\int_{\theta_1}^{\theta_2}\frac{d\theta}{cosec\,\theta}$$
$$= \frac{\mu_0 I}{4\pi R}\int_{\theta_1}^{\theta_2}\sin\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi R}[-\cos\theta]_{\theta_1}^{\theta_2}$$

$$= \frac{\mu_0 I}{4\pi R}[\cos\theta_1 - \cos\theta_2]$$

For an infinitely long conductor, $\theta_1 = 0$ and $\theta_2 = \pi$. Hence, we get

$$B = \frac{\mu_0 I}{2\pi R}$$

ii) **Magnetic Field along the Axis of a Circular Coil**

Let us consider a circular coil of radius a carrying a current I lying in the YZ plane with its centre at the origin O. The axis of the coil coincides with the X axis. Let P be a point on the axis of the coil at a distance $\vec{R}$ from O, then $\vec{R} = R\hat{\imath}$. Let the direction of the current in the coil be anti – clockwise as seen from P. Now consider a small element CD of length $d\vec{l}$, in the YZ plane and let the co – ordinates of C be $(0, y, z)$ and that of D be $(0, y + dy, z + dz)$, then

$$\overrightarrow{OC} = \vec{a} = y\hat{\jmath} + z\hat{k}$$

$$d\vec{l} = [(y + dy)\hat{\jmath} + (z + dz)\hat{k}] - [y\hat{\jmath} + z\hat{k}]$$

$$d\vec{l} = dy\hat{\jmath} + dz\hat{k}$$

Let the vector distance of the point P from the current element be $\vec{r}$, then

$$\vec{a} + \vec{r} = \vec{R}$$

$$\Rightarrow \vec{r} = \vec{R} - \vec{a}$$

$$= R\hat{\imath} - y\hat{\jmath} - z\hat{k}$$

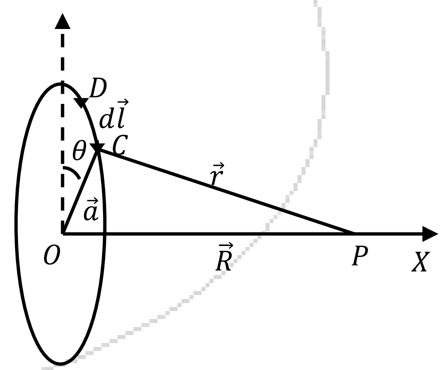
If θ is the angle that OC makes with the Y axis, then

$$y = a\cos\theta \Rightarrow dy = -a\sin\theta \; d\theta$$

$$z = a\sin\theta \Rightarrow dz = a\cos\theta \; d\theta$$

Hence, the radius vector is given by,

$$\vec{a} = a\cos\theta \; \hat{\jmath} + a\sin\theta \; \hat{k}$$



The magnetic field at the point P due to the current element $d\vec{l}$ is given by Biot Savart's law as,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I(d\vec{l} \times \vec{r})}{r^3}$$

Now,

$$d\vec{l} \times \vec{r} = (dy\hat{\jmath} + dz\hat{k}) \times (R\hat{\imath} - y\hat{\jmath} - z\hat{k})$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & dy & dz \\ R & -y & -z \end{vmatrix}$$

$$= (-zdy + ydz)\hat{\imath} + (Rdz)\hat{\jmath} + (-Rdy)\hat{k}$$

$$= (a^2 sin^2\theta \; d\theta + a^2 cos^2\theta \; d\theta)\hat{\imath} + (Ra\cos\theta)\hat{\jmath} + (Ra\sin\theta)\hat{k}$$

$$= (a^2 d\theta)\hat{\imath} + (Ra\cos\theta)\hat{\jmath} + (Ra\sin\theta)\hat{k}$$

Therefore,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I}{r^3}[(a^2 d\theta)\hat{\imath} + (Ra\cos\theta)\hat{\jmath} + (Ra\sin\theta)\hat{k}]$$

Hence, total magnetic field is given by,

$$\vec{B} = \int_0^{2\pi} d\vec{B} = \int_0^{2\pi} \frac{\mu_0}{4\pi}\frac{I}{r^3}[(a^2 d\theta)\hat{\imath} + (Ra\cos\theta)\hat{\jmath} + (Ra\sin\theta)\hat{k}]$$

$$= \frac{\mu_0}{4\pi} \frac{I}{r^3} \left[\int_0^{2\pi} (a^2 d\theta)\hat{\imath} + \int_0^{2\pi} (Ra\cos\theta)\hat{\jmath} + \int_0^{2\pi} (Ra\sin\theta)\hat{k} \right]$$

Now, $\int_0^{2\pi} d\theta = 2\pi$, $\int_0^{2\pi} \cos\theta = 0$ and $\int_0^{2\pi} \sin\theta = 0$. Therefore, the above expression gives,

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{I}{r^3} (2\pi a^2)\,\hat{\imath}$$

$$= \frac{\mu_0 I a^2}{2r^3}\,\hat{\imath}$$

Now, from the geometry of the figure, $r^2 = R^2 + a^2$. Hence,

$$\vec{B} = \frac{\mu_0 I a^2}{2(R^2 + a^2)^{3/2}}\,\hat{\imath}$$

For a coil of n turns, the magnetic field is given as,

$$\vec{B} = \frac{\mu_0 n I a^2}{2(R^2 + a^2)^{3/2}}\,\hat{\imath}$$

Hence, when the current carrying coil lies in the YZ plane and its axis coincides with the X axis, the magnetic field is along the $+X$ direction for an anti – clockwise current.

a. At the centre of the coil, $R = 0$. Hence, the magnetic field is given as,

$$\vec{B} = \frac{\mu_0 n I}{2a}\,\hat{\imath}$$

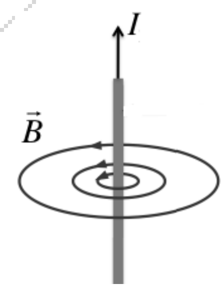
b. For $R \gg a$, the magnetic field is given as,

$$\vec{B} = \frac{\mu_0 n I a^2}{2R^3}\,\hat{\imath}$$

The field decreases as we move away from the centre of the coil along its axis. It is maximum at the centre and decreases symmetrically on both sides along $+X$ and $-X$ direction. The variation of magnetic field with distance along the axis from the centre is as shown in figure.

Ampere's Circuital Law

It states that the line integral of the magnetic induction around a closed path is equal to the total current enclosed by the path multiplied by the permeability of the free space.



If a magnetic field $\vec{B}$ encloses a current I flowing through a straight conducting wire in a direction perpendicular to the plane of the closed path and $d\vec{l}$, a vector element of the path, then

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

where, $\oint \vec{B} \cdot d\vec{l}$ is the line integral of the magnetic field for the closed path.

Consider an infinitely long straight wire carrying a current I perpendicular to the plane of paper flowing outwards i.e. along the $+Z$ direction. At any point at a distance r from the wire, the magnetic field is given by $\frac{\mu_0 I}{2\pi r}$. The sense of the direction of magnetic field will be given by right hand screw rule. The magnetic lines will lie in the XY plane and would be anti – clockwise in direction.

If we draw a circle with the point where the wire leaves the paper as the centre and radius r, then the circumference of the circle is a line of constant magnetic induction $\vec{B} = \frac{\mu_0 I}{2\pi r}$ and its direction is along the tangent to the circle in the plane of the paper in the anti – clockwise direction.

A small element $d\vec{l}$ of the circular line of constant magnetic induction is also tangential to the circle and hence $\vec{B}$ and $d\vec{l}$ are parallel. Hence, we get,

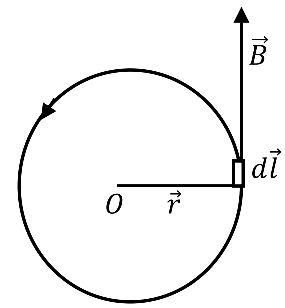
$$\vec{B} . d\vec{l} = B\,dl\cos\theta = B\,dl \qquad ,\theta = 0$$

The value of the line integral for the whole closed path is given by,

$$\oint \vec{B}\,.\,d\vec{l} = B\oint d\vec{l} = B(2\pi r) = \frac{\mu_0 I}{2\pi r}(2\pi r)$$

$$\oint \vec{B}\,.\,d\vec{l} = \mu_0 I$$

The relation between current density $\vec{J}$ for current I is given by,



$$I = \iint_S \vec{J}\,.\,d\vec{S}$$

Therefore, Ampere's law takes the form,

$$\oint \vec{B}\,.\,d\vec{l} = \mu_0 \iint \vec{J}\,.\,d\vec{S}$$

But according to, Stoke's theorem of vectors,

$$\oint \vec{B}\,.\,d\vec{l} = \iint_S (Curl\,\vec{B})\,.\,d\vec{S} = \iint_S (\vec{\nabla}\times\vec{B})\,.\,d\vec{S}$$

Hence,

$$\iint_S (\vec{\nabla}\times\vec{B})\,.\,d\vec{S} = \mu_0 \iint \vec{J}\,.\,d\vec{S}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

This is the Ampere's circuital law in the differential form. It is one of the Maxwell's laws of electrodynamics.

<u>Independence of the shape of the path</u>

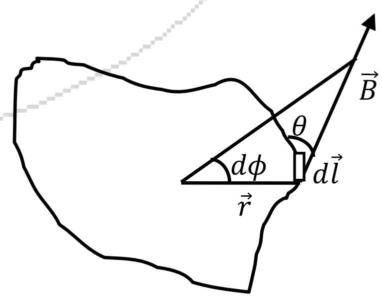
Let's consider any arbitrary closed path which encircles the wire carrying current. As the wire carries a steady current in the $+Z$ direction, the magnetic field due to it lies entirely in XY plane. Hence, the closed path also lies in the XY plane. Hence,

$$\vec{B}.d\vec{l} = B\,dl\cos\theta$$

where, $dl\cos\theta$ is the component of the line element $d\vec{l}$ along the direction of $\vec{B}$, θ being the angle between the vector $d\vec{l}$ and vector $\vec{B}$ which is perpendicular to the radius vector $\vec{r}$. Thus $dl\cos\theta$ is also perpendicular to the radius vector $\vec{r}$. If $d\phi$ is the angle subtended by $d\vec{l}$ at the wire, then

$$d\phi = \frac{dl\cos\theta}{r}$$

Hence, Ampere's circuital law is given as,



$$\vec{B}.d\vec{l} = B\,dl\cos\theta = B\,r\,d\phi = \frac{\mu_0 I}{2\pi r}r\,d\phi = \frac{\mu_0 I}{2\pi}d\phi$$

$$\oint \vec{B}\,.\,d\vec{l} = \frac{\mu_0 I}{2\pi}\oint d\phi = \frac{\mu_0 I}{2\pi}(2\pi)$$

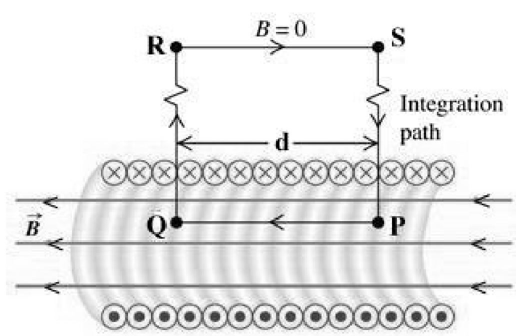
$$\oint \vec{B}\,.\,d\vec{l} = \mu_0 I$$

Thus it is seen that the line integral of the magnetic induction vector over a closed path is independent of the nature of the path.

<u>Applications</u>

i) **<u>Magnetic Field for a Long Solenoid</u>**

A solenoid is a long wire wound in a close – packed helix and carries a current. Let us consider a solenoid of n turns per unit length having length L and radius r with the condition, $L \gg r$. Let it carry a current I. The behaviour of the field for a single turn is the same as that of a straight



wire. The total field for all the turns is the vector sum of the fields set up by all the turns. The field of a solenoid is uniform on its cross – section in the middle region of the solenoid.

Let us consider a rectangular path $PQRS$. The integral of the magnetic field over the closed path $PQRS$ is the sum of the four integrals, one for each path of the rectangle.

$$\oint_{PQRS} \vec{B}.d\vec{l} = \int_P^Q \vec{B}.d\vec{l} + \int_Q^R \vec{B}.d\vec{l} + \int_R^S \vec{B}.d\vec{l} + \int_S^P \vec{B}.d\vec{l}$$

a. Outside the solenoid, along the path RS, no current is enclosed and hence the magnetic field is zero. Hence, the line integral is given as,

$$\int_R^S \vec{B}.d\vec{l} = 0$$

b. Along the path QR and SP, the value of the line integral is zero for the points outside the solenoid. For the points inside the solenoid, the magnetic field is perpendicular to the line element $d\vec{l}$. Hence, $\vec{B}.d\vec{l} = B\ dl \cos\theta = B\ dl \cos\left(\frac{\pi}{2}\right) = 0$. Hence,

$$\int_Q^R \vec{B}.d\vec{l} = \int_S^P \vec{B}.d\vec{l} = 0$$

c. Inside the solenoid, along the path PQ, magnetic field is constant and along PQ i.e. $\vec{B}$ is parallel to $d\vec{l}$. If $PQ = d$ and the number of turns in $PQ = nd$, then

$$\int_P^Q \vec{B}.d\vec{l} = B\int_P^Q d\vec{l} = Bd$$

Hence, Ampere circuital law gives,

$$\oint_{PQRS} \vec{B}.d\vec{l} = \int_P^Q \vec{B}.d\vec{l} = \mu_0(nd)I$$

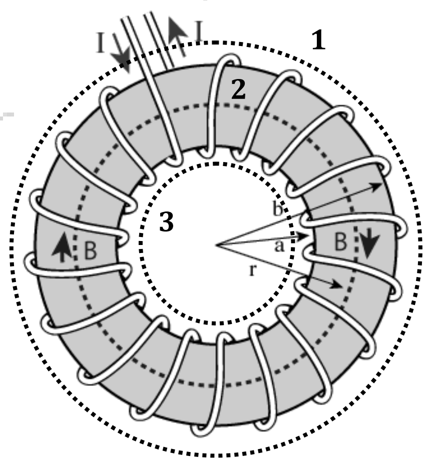
$$Bd = \mu_0(nd)I$$

$$B = \mu_0 nI$$

ii) <u>Magnetic Field for a Torroid</u>

A torroid may be visualized as a doughnut with wire wound uniformly around the doughnut. The torroid may be considered as a solenoid that has been bent into a circle with the ends joined. Since, the axis of the resulting bent solenoid is circular, the lines of $\vec{B}$ are also circular. Let us consider paths,

a. Outside the torroid
b. Inside the torroid



Outside the torroid, no current is enclosed hence the magnetic field is zero. Hence,

$$\oint_1 \vec{B}.d\vec{l} = \oint_3 \vec{B}.d\vec{l} = 0$$

Inside the torroid, the current is enclosed. Hence, for mean radius r of the torroid,

$$\oint_2 \vec{B}.d\vec{l} = B\oint_2 d\vec{l} = B(2\pi r)$$

Hence,

$$\oint \vec{B}.d\vec{l} = B(2\pi r) = \mu_0 nI$$

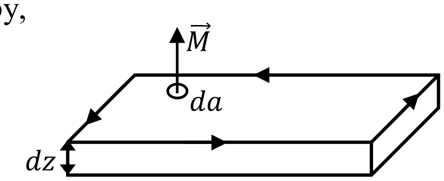
$$B = \frac{\mu_0 nI}{2\pi r}$$

Magnetization Current

The electron in an atom has a magnetic dipole moment due to its orbital motion around the nucleus. It also has a moment owing to its spin. Due to this fact certain atoms and molecules have a net magnetic moment.

Let us consider a small rectangular piece of uniformly magnetized material of magnetization $\vec{M}$ having surface area da and thickness dz, then its magnetic moment is given by,

$$\vec{m} = \vec{M}\, dV = \vec{M}\, dadz$$



Also for a current loop, magnetic moment is given by,

$$\vec{m} = current \times area\ of\ loop$$
$$= I\, da$$

Hence,

$$I\, da = \vec{M}\, dadz$$
$$I = M\, dz$$

This current is due to the orbital motion of the electron around the nucleus and hence is a bound current. This is called the Magnetization current.

A magnetic dipole moment of moment $\vec{M}\, dadz$ is equivalent to a loop of area da through the boundary of which flows the current Mdz in the anticlockwise direction. For two identical pieces kept side by side, the currents in the boundary where they touch each other are equal and opposite and hence they cancel out. Thus for these two adjoining pieces the current is $I = M\, dz$. Extending this to the entire sample, the current is given as, $I = M\, dz$.

By placing slabs of thickness dz one above the other, we can obtain a magnetized material of any dimension. The total current is given as, $I = \sum M\, dz = M \sum dz = Mz$. The surface current density through unit length is given by,

$$J_z = \frac{I}{z} = M$$

Let us consider the variation of M_z along the Y axis, the current density is given by,

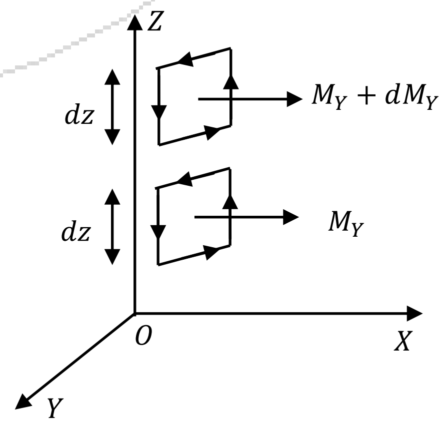
$$J_{x_1} = \frac{\partial M_z}{\partial y} \quad along\ X\ direction$$

Similarly if variation of M_y along Z axis is considered, the current density is given by,

$$J_{x_2} = -\frac{\partial M_y}{\partial z} \quad along\ X\ direction$$

Hence, total contribution to current density along X direction is given as,

$$J_x = J_{x_1} + J_{x_2}$$
$$= \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$



On the same lines, the value of J_y and J_z is given as,

$$J_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}$$
$$J_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}$$

Hence, the current density vector is given as,

$$\vec{J} = J_x\hat{\imath} + J_y\hat{\jmath} + J_z\hat{k}$$
$$= \left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right)\hat{k}$$
$$= \vec{\nabla} \times \vec{M} = Curl\ \vec{M}$$

Since, the current density is due to the bound currents, we get

$$Curl\ \vec{M} = \vec{J}_{bound}$$

If the material is magnetized uniformly then $\vec{M}$ is constant and hence $\vec{\nabla} \times \vec{M} = 0$. Therefore,

$$\vec{J}_{bound} = 0$$

The total current density is given by,

$$\vec{J}_{total} = \vec{J}_{bound} + \vec{J}_{free}$$

Hence, the expression for Ampere's circuital law, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$, can be written as

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{total} = \mu_0 (\vec{J}_{bound} + \vec{J}_{free})$$

$$= \mu_0 \vec{J}_{free} + \vec{\nabla} \times \vec{M}$$

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_{free}$$

This new quantity $\frac{\vec{B}}{\mu_0} - \vec{M}$ is called the magnetic field intensity vector and is denoted by $\vec{H}$. Thus,

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free}$$

This is known as the differential form of Ampere's law in the presence of magnetic material.

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