Gravitation

Kepler's Law

BSc – I SEM – II (UNIT I)

Contents

Newton's Law of Gravitation

Every particle of matter in the universe attracts every other particle with a force whose magnitude is,

i) directly proportional to the product of their masses.
ii) inversely proportional to the square of the distance between them.

If m_1 and m_2 are the masses of two bodies and r is the distance between them, then the force of attraction, F, is given as,

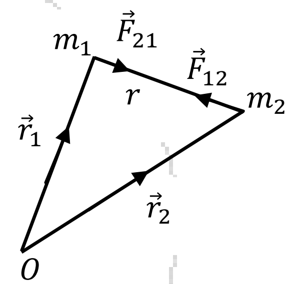
$$F \propto -\frac{m_1 m_2}{r^2}$$

The negative sign in the above signifies that gravitational force is always attracting and directed towards the centre of mass for the concerned masses. The constant of the above equation is given as G which is called the Gravitational constant. It's value is $6.67 \times 10^{-11} Nm^2/Kg^2$. The dimension of G is given as, $[M^{-1}L^3T^{-2}]$. Hence, the gravitational force is given as,

$$\boldsymbol{F = -\frac{G m_1 m_2}{r^2}}$$

Vector Representation of Newton's Law of Gravitation

Let two masses m_1 and m_2 be at two points whose position vectors are $\vec{r}_1$ and $\vec{r}_2$ with respect to origin O. The gravitational force on mass m_1 due to mass m_2, $\vec{F}_{12}$ is given as,



$$\vec{F}_{12} = -\frac{G m_1 m_2}{\vec{r}_{21}{}^2}\hat{r}_{21}$$

where, $\vec{r}_{21}$ is the direction vector in the direction of mass m_2 to m_1 and $\hat{r}_{21}$ is the unit vector in the direction of $\vec{r}_{21}$.

Similarly, the gravitational force on mass m_2 due to mass m_1, $\vec{F}_{21}$ is given as,

$$\vec{F}_{21} = -\frac{G m_1 m_2}{\vec{r}_{12}{}^2}\hat{r}_{12}$$

where, $\vec{r}_{12}$ is the direction vector in the direction of mass m_1 to m_2 and $\hat{r}_{12}$ is the unit vector in the direction of $\vec{r}_{12}$. By laws of vectors, we know that $\vec{r}_{12} = -\vec{r}_{21}$ and $|\vec{r}_{12}| = |\vec{r}_{21}|$. Therefore, $\hat{r}_{12} = -\hat{r}_{21}$ and $\vec{r}_{21}{}^2 = \vec{r}_{12}{}^2 = r^2$. Hence, we get,

$$\boldsymbol{\vec{F}_{12}} = \frac{G m_1 m_2}{\vec{r}_{12}{}^2}\hat{r}_{12} = \boldsymbol{-\vec{F}_{21}}$$

Thus, the two forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are equal in magnitude but opposite in direction.

Characteristics of Newton's Law of Gravitation

i) Gravitational forces between two particles form a pair of action and reaction according to the Newton's third law of motion.
ii) The force is a central force exerted by one particle on another along a line joining the two particles.
iii) It is independent of other bodies present between concerned bodies.
iv) For spherical bodies of uniform density, mass can be considered to be concentrated at the centre of mass.
v) The universal constant of gravitation, G, does not depend on the shape, size, nature of bodies nor does it depend on the medium in between them. It remains constant at all places at all times.
vi) It is the weakest force among the four forces but is considerable for heavy bodies.
vii) It is a long range force.

INTENSITY OF GRAVITATIONAL FIELD

The space around a body within which its gravitational force of attraction can be experienced is called the gravitational field.

The intensity of the gravitational field at a point due to a body is defined as the force exerted by the body on a unit mass placed at that point in the field. If F is the force of gravitation due to a body of mass M on a body of unit mass m placed at a distance r from it then the intensity of the gravitational field, E, is given as,

$$\vec{E} = \vec{F}/m$$

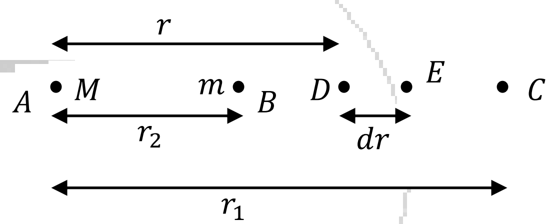
$$\Rightarrow \vec{E} = {-\frac{GMm}{r^2}\hat{r}}\Big/_{m}$$

$$\Rightarrow \boldsymbol{\vec{E} = -\frac{GM}{r^2}\hat{r}}$$

The unit of intensity of gravitational field is N/Kg and the dimension is $[LT^{-2}]$.

GRAVITATIONAL POTENTIAL ENERGY

For a particle of mass m situated at a point in the gravitational field of a body of mass M, gravitational potential energy is defined as the work done by the gravitational force exerted on it by the body of mass M in moving the particle from infinity to that point.



Let a particle of mass M be kept fixed at a point A and another particle of mass m is moved from C to B. Initially the distance be $AC = r_1$ and the finally it be $AB = r_2$.

Consider a small displacement from E to D, when the distance changes from $AE = r + dr$ to $AD = r$.

The gravitational force on mass m due to mass M is given as,

$$F = -\frac{GMm}{r^2}$$

The work done in moving the mass from E to D along the force is given as,

$$dW = F.dr = -\frac{GMm}{r^2}dr$$

This work done is stored as the potential energy, U, at that point. Potential energy of a system corresponding to conservative force acting from i to f, is given as,

$$U = -\int_i^f dW = -\int_i^f F.dr$$

Hence, in our case, we get

$$dU = \int_{r_1}^{r_2} \frac{GMm}{r^2}dr$$

$$U = GMm\int_{r_1}^{r_2}\frac{dr}{r^2} = GMm\left[-\frac{1}{r}\right]_{r_1}^{r_2} = GMm\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Let us now take $r_1 = \infty$ and $r_2 = r$. Then we get

$$U = GMm\left[\frac{1}{\infty} - \frac{1}{r}\right]$$

$$\boldsymbol{U = -\frac{GMm}{r}}$$

The unit for gravitational potential energy is Joules (J).

Gravitational Potential

The work done in moving a unit mass from infinity to any point in the gravitational field of a body of mass M is called as the gravitational potential at that point due the body.

It may also be defined as the potential energy of a unit mass placed at any point in the gravitational field of a body of mass M. Hence gravitational potential, V, is given as

$$V = U/m$$

The unit for gravitational potential is J/Kg. The dimension is, $[L^2T^{-2}]$

Relation between Gravitational Field and Gravitational Potential

Let a particle of mass m be kept in the gravitational field of a body of mass M. Let F be the gravitational force of attraction exerted by the mass M on mass m. Hence, the work done to displace the mass m through a distance δx is given as,

$$\delta W = F\delta x$$

Hence, the corresponding change in potential energy is given by,

$$\delta U = -\delta W = -F\delta x$$

$$\frac{\delta U}{\delta x} = -F$$

$$\lim_{\delta x \to 0} \frac{\delta U}{\delta x} = \frac{dU}{dx} = -F$$

Now, we know that potential energy is given by $U = mV$ and force is given by $F = mE$, where V is the gravitational potential and E is the intensity of gravitational field.

$$\frac{dV}{dx} = -E$$

$$\boldsymbol{E = -\frac{dV}{dx}}$$

In three dimensional space, the above equation can be written as,

$$\boldsymbol{E = -\nabla V}$$

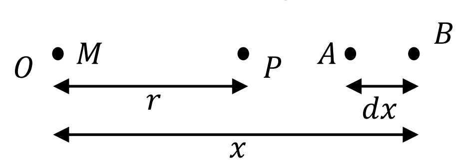
Thus, intensity of gravitational field is the space rate of change of gravitational potential.

Calculation of Gravitational Potential and Intensity of Gravitational Field

1. Due to a Point Mass

Let a particle of mass M be kept at point O and P be a point at a distance r from O. Let points A and B be at distances $x - dx$ and x respectively. Intensity of gravitational field, E, at B is given as,

$$E = \frac{-GM}{x^2}$$



Hence, the work done by the gravitational force in moving a unit mass from B to A is given as,

$$dW = Fdx$$

In terms of gravitational potential and intensity of gravitational field, above equation can be written as,

$$dV = -Edx$$

We have, $V(\infty) = 0$ and $V(r) = V_r$. Hence, we get

$$\int_0^{V_r} dV = -\int_\infty^r E dx$$

$$V_r = GM \int_\infty^r \frac{dx}{x^2}$$

$$V_r = GM \left[-\frac{1}{x}\right]_\infty^r$$

$$\boldsymbol{V_r = \frac{-GM}{r}}$$

2. Due to a Uniform Thin Spherical Shell

Let us consider a uniform thin spherical shell of mass M and radius R. We have to calculate the potential due to this shell at a point P. Let the centre of shell be O and $OP = r$.

Let us draw a radius OA making an angle θ. Keeping the $\angle AOP = \theta$ fixed, let us rotate this radius about OP. The point A traces a circle on the surface of the shell. Let us now consider another radius OB at an angle $\theta + d\theta$ and draw a similar circle about OP. The part of the spherical shell between these two circles can be treated as a ring.

The width of the ring is given as $Rd\theta$ and the radius is given as $AE = R\sin\theta$. The area of the ring is hence given as,

$$= (2\pi R\sin\theta)(Rd\theta) = 2\pi R^2 \sin\theta \ d\theta$$

Total surface area of the spherical shell is $4\pi R^2$. Hence, mass of the enclosed ring is given as,

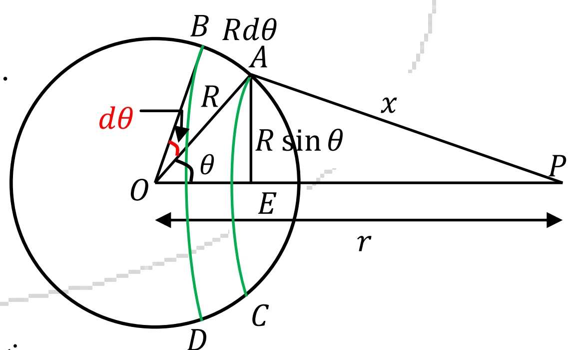
$$dm = \frac{M}{4\pi R^2} 2\pi R^2 \sin\theta \ d\theta = \frac{M}{2}\sin\theta \ d\theta$$

Let $AP = x$.

Then by geometry, $x^2 = R^2 + r^2 - 2rR\cos\theta$.

On differentiating we get, $2xdx = 2rR\sin\theta \ d\theta$

$$\Rightarrow \sin\theta \ d\theta = \frac{x \, dx}{rR}$$



Then the mass of the enclosed ring is given as,

$$dm = \frac{M}{2}\frac{x \, dx}{rR}$$

The gravitational potential at P for this elemental ring is given as,

$$dV = -G\frac{dm}{x}$$

$$dV = -\frac{GM}{2rR}dx$$

As we vary θ from 0 to π, the rings formed on the shell cover up the whole shell. The potential due to the whole shell can be obtained by integrating dV within the limits $\theta = 0$ to $\theta = \pi$.

Case 1. *P* is outside the shell ($r > R$)

As figure shows, when $\theta = 0, x = r - R$ and when $\theta = \pi, x = r + R$. Hence, we get

$$V = \int dV = -\frac{GM}{2rR}\int_{r-R}^{r+R} dx = -\frac{GM}{2rR}[r + R - r + R]$$

$$\boldsymbol{V = -\frac{GM}{r}}$$

The intensity of gravitational field is given as the space rate of change of potential. Hence,

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right)$$

$$\boldsymbol{E = -\frac{GM}{r^2}}$$

The gravitational force acting on a particle of mass m at P is given as,

$$\boldsymbol{F} = mE = -\boldsymbol{\frac{GmM}{r^2}}$$

Case 2. **P is on the surface of the shell ($r = R$)**

The gravitational potential, intensity of gravitational field and gravitational force is given as,

$$\boldsymbol{V = -\frac{GM}{R}}$$

$$\boldsymbol{E = -\frac{GM}{R^2}}$$

$$\boldsymbol{F} = mE = -\boldsymbol{\frac{GmM}{R^2}}$$

Case 3. **P is inside the shell ($r < R$)**

As figure shows, when $\theta = 0, x = R - r$ and when $\theta = \pi, x = R + r$. Hence, we get

$$V = \int dV = -\frac{GM}{2rR}\int_{R-r}^{R+r} dx$$

$$V = -\frac{GM}{2rR}[R + r - R + r]$$

$$\boldsymbol{V = -\frac{GM}{R}}$$

The intensity of gravitational field is given as the space rate of change of potential. Hence,

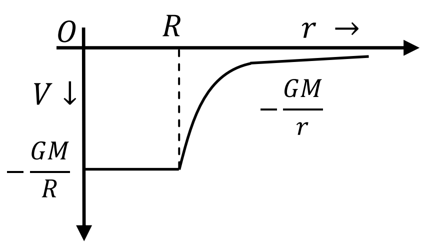
$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{R}\right)$$

$$\boldsymbol{E = 0}$$

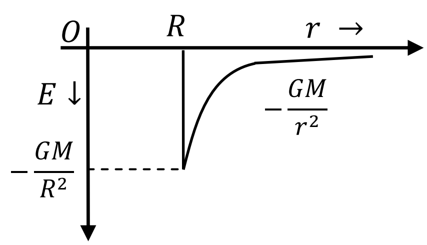
The gravitational force acting on a particle of mass m at P is given as,

$$\boldsymbol{F} = mE = \boldsymbol{0}$$

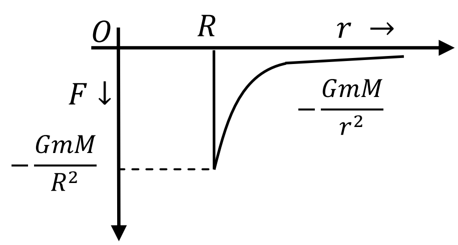
The figure shows the variations of gravitational potential, intensity of gravitational field and gravitational force with changing values of distance r.



Gravitational Potential



Intensity of Gravitational Field



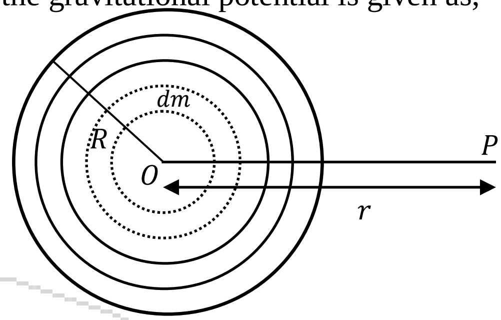
Gravitational Force

3. Due to a Uniform Solid Sphere

Let us consider a solid sphere of uniform density with radius R and mass M. Let P be a point where the gravitational potential is to be calculated. Let the distance from centre of sphere O to the point P be r.

Case 1. $\boldsymbol{P}$ is outside the shell ($\boldsymbol{r} > R$)

Let us consider that the sphere is composed of a large number of concentric shells with masses $m_1, m_2, \ldots$ For a mass of a shell, let us say dm, the gravitational potential is given as,



$$dV = -\frac{G\,dm}{r}$$

Hence, the potential due to the entire sphere is given as,

$$V = \int dV = -\frac{G}{r}\int dm$$

$$\boldsymbol{V = -\frac{GM}{r}}$$

The gravitational potential due to a uniform sphere at an external point is same as that due to a single particle of mass placed at its centre.

The intensity of gravitational field is given as the space rate of change of potential. Hence,

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right)$$

$$\boldsymbol{E = -\frac{GM}{r^2}}$$

The gravitational force acting on a particle of mass m at P is given as,

$$\boldsymbol{F} = mE = -\boldsymbol{\frac{GmM}{r^2}}$$

Case 2. $\boldsymbol{P}$ is on the surface of the shell ($\boldsymbol{r = R}$)

The gravitational potential, intensity of gravitational field and gravitational force is given as,

$$\boldsymbol{V = -\frac{GM}{R}}$$

$$\boldsymbol{E = -\frac{GM}{R^2}}$$

$$\boldsymbol{F} = mE = -\boldsymbol{\frac{GmM}{R^2}}$$

Case 3. P is inside the shell ($\boldsymbol{r} < R$)

Let us draw two spheres of radii x and $x + dx$. These two spheres will enclose a spherical shell of volume $4\pi x^2 dx$. The volume of the given sphere is $\frac{4}{3}\pi R^3$. As the sphere is uniform, the mass of the shell is given as,

$$dm = \frac{M}{\frac{4}{3}\pi R^3} 4\pi x^2 dx = \frac{3M}{R^3} x^2 dx$$

Now, let us draw a sphere with $OP = r$ as the radius. Then the mass enclosed in this sphere is given as,

$$M' = \frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = \frac{M}{R^3}r^2$$

At the point P, the gravitational potential will be due to two potentials,

i) V_1 due to the mass enclosed inside the sphere of radius $OP = r$. This is given as,

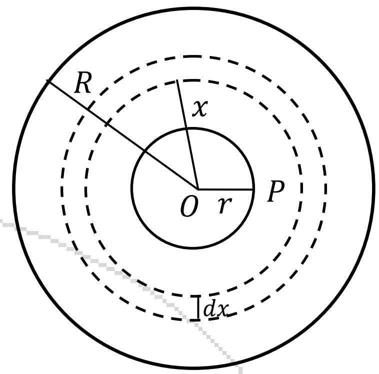
$$V_1 = \int dV_1 = -\int \frac{G\,dm}{r} = -\frac{G}{r}\int dm = -\frac{G}{r}M'$$

$$V_1 = -\frac{GMr^2}{R^3}$$

ii) V_2 due to the mass enclosed outside the sphere of radius $OP = r$. This is given as,

$$V_2 = \int dV_2 = -\int_r^R \frac{G\,dm}{x} = -\frac{3GM}{R^3}\int x dx$$

$$V_2 = -\frac{3GM}{R^3}[x^2]_r^R = -\frac{3GM}{R^3}[R^2 - r^2]$$



Hence, total potential is given by,

$$V = V_1 + V_2$$

$$V = -\frac{GMr^2}{R^3} - \frac{3GM}{R^3}[R^2 - r^2]$$

$$\boldsymbol{V = -\frac{3GM}{2R^3}[3R^2 - r^2]}$$

The intensity of gravitational field is given as the space rate of change of potential. Hence,

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{3GM}{2R^3}[3R^2 - r^2]\right)$$

$$\boldsymbol{E = -\frac{GM}{R^3}r}$$

The gravitational force acting on a particle of mass m at P is given as,

$$\boldsymbol{F} = mE = -\boldsymbol{\frac{GmM}{R^3}r}$$

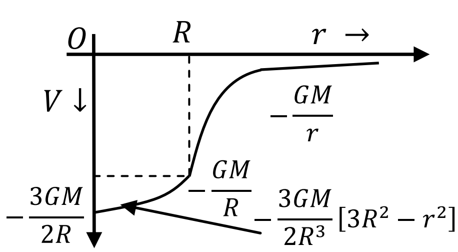
At the centre of the sphere, $r = 0$. Hence, the values are given as,

$$\boldsymbol{V = -\frac{3GM}{2R}}$$

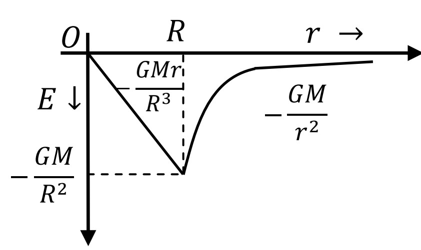
$$\boldsymbol{E = 0}$$

$$\boldsymbol{F} = mE = \boldsymbol{0}$$

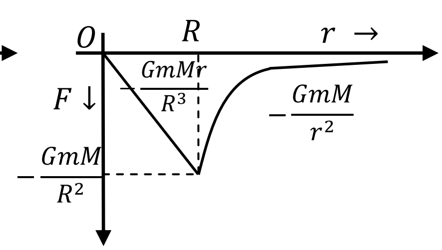
The figure shows the variations of gravitational potential, intensity of gravitational field and gravitational force with changing values of distance r.



Gravitational Potential



Intensity of Gravitational Field



Gravitational Force

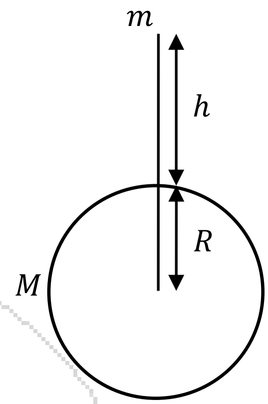
Acceleration Due to Gravity

It is observed that when a body is allowed to fall freely under gravity, its acceleration remains constant. The acceleration produced in a free falling body due to the gravitational pull of the earth is called acceleration due to gravity. It is represented by g. It's value on the surface of the earth is 9.8 m/s^2.

Let earth have mass M and radius R. According to Newton's law of gravitation, gravitational force on a mass m situated at a height h from the surface of the earth is given as,

$$F = \frac{GMm}{(R+h)^2}$$

Also according to the Newton's second law, the force on a falling object is given as,



$$Force = mass \times acceleration$$

$$F = mg_h$$

where, g_h is the acceleration due to gravity of the object situated at height h.
Equating both the equations, we get

$$\boldsymbol{g_h = \frac{GM}{(R+h)^2}}$$

On the surface of earth $h = 0$, acceleration due to gravity is given as,

$$\boldsymbol{g = \frac{GM}{R^2}}$$

Hence,

$$\boldsymbol{g_h = \frac{gR^2}{(R+h)^2}}$$

Effects on Acceleration due to Gravity

i. **Height above the surface of earth**

For height above the surface of earth, acceleration due to gravity is given as,

$$g_h = \frac{GM}{(R+h)^2}$$

$$g_h = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2} = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

If $h \ll R$, then neglecting the higher powers of h/R, we get

$$g_h = g\left(1+\frac{h}{R}\right)^{-2}$$

$$\boldsymbol{g_h = g\left(1-\frac{2h}{R}\right)}$$

ii. **Depth below the surface of earth**

Inside the surface of earth, the gravitational force is given as,

$$F = \frac{GmM}{R^3}r$$

In our case for any depth d, $r = R - d$. Hence, we get

$$F = \frac{GmM}{R^3}(R-d)$$

Hence, acceleration due to gravity is given as,

$$g_h = \frac{F}{m} = \frac{GM}{R^2}\left(\frac{R-d}{R}\right)$$

$$\boldsymbol{g_h = g\left(1 - \frac{d}{R}\right)}$$

iii. **Shape of earth**

Since earth is an oblate spheroid in shape, equatorial radius is about 21 kilometres more than the polar radius. Hence, gravitational force of attraction at the poles is more than the equator. Thus the value of acceleration due to gravity at the poles is more than that at the equator.

iv. **Effect of rotation**

The value of acceleration due to gravity owing to the earth's rotation is given by the equation,

$$g' = g - R\omega^2 cos^2\,\phi$$

where, ϕ is the latitude and ω is the angular velocity of earth's rotation. Thus the value of acceleration due to gravity at the poles is more than that at the equator.

v. **Non – Uniformity of Earth**

The earth is not a uniformly dense object. It is composed of minerals, metals, water, oil etc. The surface features vary from mountains to deep seas. These non – uniformities in the mass distribution affect the value of acceleration due to gravity locally.

Gravitational Self Energy

Every body possesses gravitational self – energy due to its mass. The self – energy of a body is equal to the work done by gravitational force in bringing infinite small masses from infinity to the point where the body is formed. The gravitational self – energy of a body is negative because the gravitational force is negative.

Suppose a body consists of N particles each of mass m_1, m_2, ... , m_N and let distance between masses m_i and m_j be r_{ij}, then, gravitational self energy is given by,

$$\boldsymbol{U = -\frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j}{r_{ij}}}$$

In the gravitational field of m_1, the potential energy of m_2, m_3, ... , m_N is given by,

$$U_1 = -G\left[\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \cdots + \frac{m_1 m_N}{r_{1N}}\right]$$

$$U_1 = -Gm_1\left[\frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} + \cdots + \frac{m_N}{r_{1N}}\right]$$

$$U_1 = -Gm_1\sum_{\substack{j=1\\ j\neq 1}}^{N}\frac{m_j}{r_{1j}}$$

Similarly, in the gravitational field of m_2, the potential energy of m_1, m_3, ... , m_N is given by,

$$U_2 = -Gm_2\sum_{\substack{j=1\\ j\neq 2}}^{N}\frac{m_j}{r_{2j}}$$

The gravitational self energy will be the sum of the energies, U_1, U_2, ... , U_N. Since each term in the summation occurs twice, we divide the sum by 2.

$$U = \frac{U_1 + U_2 + \cdots + U_N}{2}$$

$$U = -\frac{G}{2}\left[m_1 \sum_{\substack{j=1 \\ j\neq 1}}^{N} \frac{m_j}{r_{1j}} + m_2 \sum_{\substack{j=1 \\ j\neq 2}}^{N} \frac{m_j}{r_{2j}} + \cdots + m_N \sum_{\substack{j=1 \\ j\neq N}}^{N} \frac{m_j}{r_{Nj}}\right]$$

$$U = -\frac{G}{2}\left[\sum_{i=1}^{N} m_1 \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{r_{ij}}\right]$$

$$\boldsymbol{U = -\frac{G}{2}\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_i m_j}{r_{ij}}}$$

Gravitational Self Energy of a Galaxy

There are a huge number of galaxies in the universe each containing a large number of stars. Our own galaxy, Milky Way, is supposed to contain about 1.6×10^{11} stars. All the objects in a galaxy are bounded by the gravitational force. Similar gravitational force exists between different galaxies as well.

Let a galaxy consists of N stars each of mass M_1, M_2, ... , M_N and let distance between two stars having masses M_i and M_j be R_{ij}, then, gravitational self energy is given by,

$$U = -\frac{G}{2}\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{M_i M_j}{R_{ij}}$$

Substituting $R_{ij} = R$ for all i, j, we get

$$U = -\frac{G}{2R}\left[\sum_{i=1}^{N} M_i \sum_{\substack{j=1 \\ j\neq i}}^{N} M_j\right]$$

$$U = -\frac{G}{2R}\left[M_1 \sum_{\substack{j=1 \\ j\neq 1}}^{N} M_j + M_2 \sum_{\substack{j=1 \\ j\neq 2}}^{N} M_j + \cdots + M_N \sum_{\substack{j=1 \\ j\neq N}}^{N} M_j\right]$$

$$U = -\frac{G}{2R}[M_1[M_2 + M_3 + \cdots + M_N] + M_2[M_1 + M_2 + \cdots + M_N] + \cdots + M_N[M_1 + M_2 + \cdots + M_{N-1}]]$$

Let $M_1 = M_2 = \cdots = M_N = M$. Also there are $N - 1$ terms in each bracket. Hence, we get

$$U = -\frac{G}{2R}[M[(N-1)M] + M[(N-1)M] + \cdots + M[(N-1)M]]$$

$$U = -\frac{G}{2R}M^2 N(N-1)$$

As $N > 1$, $(N - 1)$ is taken to be N. Hence, gravitational self energy of a galaxy is found to be,

$$\boldsymbol{U = -\frac{GN^2M^2}{2R}}$$

GAUSS'S THEOREM OF GRAVITATION

The gravitational flux ϕ through any closed surface enclosing total mass M is given as,

$$\boldsymbol{\phi = -4\pi GM}$$

This is called as Gauss's theorem of gravitation.

From any point mass m, the gravitational field, $\vec{E}$, at a distance r is given as,

$$E = -\frac{Gm}{r^2}$$

The gravitational flux is due to the normal component of gravitational field, $\vec{E}$ i.e. $E\cos\theta$. Hence the flux $d\phi$ through an area element dA is given as,

$$d\phi = -\frac{Gm}{r^2}\cos\theta dA = \vec{E}.\overrightarrow{dA} = \hat{n}.EdA$$

Hence, the total gravitational flux enclosed by the closed surface is given as,

$$\phi = \int_S d\phi = \int_S \vec{E}.\overrightarrow{dA}$$

$$\phi = \int_S \overrightarrow{E_1}.\overrightarrow{dA} + \int_S \overrightarrow{E_2}.\overrightarrow{dA} + \cdots$$

where, $\overrightarrow{E_1}$, $\overrightarrow{E_2}$, ... are due to masses M_1, M_2, ... Hence, we get

$$\phi = \left[\left(-\frac{GM_1}{r^2}\right) + \left(-\frac{GM_2}{r^2}\right) + \cdots\right] 4\pi r^2$$

$$\phi = -4\pi G(M_1 + M_2 + \cdots)$$

$$\phi = -4\pi GM$$

where, M is the total mass enclosed by the surface. The above can also be written as,

$$\int_S \vec{E}.\overrightarrow{dA} = -4\pi GM$$

This is called the integral form of Gauss's law.

For a continuous mass distribution, gravitational flux, ϕ, can be written as

$$\phi = -\int_m 4\pi G dm = -\int_V 4\pi G\rho\, dV$$

where, ρ is the density of the mass distribution and V is the volume enclosed by the surface. Thus, $dm = \rho dV$.

Hence, we get

$$\oint \hat{n}.EdA = -\int_V 4\pi G\rho\, dV$$

Applying Gauss's divergence theorem to the LHS of the equation, we get

$$\int_V \vec{\nabla}.\vec{E}\, dV = -\int_V 4\pi G\rho\, dV$$

$$-\int_V \left(\vec{\nabla}.\vec{E} + 4\pi G\rho\right) dV = 0$$

Since, this is true for any volume, we get

$$\vec{\nabla}.\vec{E} + 4\pi G\rho = 0$$

$$\vec{\nabla}.\vec{E} = -4\pi G\rho$$

This is called the differential form of Gauss's law.

Now, we can define a gravitational field, $\vec{E}$, in terms of gravitational potential V as,

$$\vec{E} = -\vec{\nabla} V$$

Hence, we get

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.-(\vec{\nabla} V) = -4\pi G\rho$$

$$\nabla^2 V = 4\pi G\rho$$

This is known as the Poisson's equation of gravitation.

Kepler's Laws of Planetary Motion

Kepler's three laws of planetary motion are,

Law 1. Each planet moves in an elliptical orbit with the Sun at one of its foci.

The eccentricity of the orbit of a particle moving under a central force is given by,

$$\varepsilon^2 = 1 + \frac{2EJ^2}{mc^2}$$

where, E is the total energy of the particle, J is the angular momentum, m is the mass of the particle and $c = GmM$ is the force constant, M being the mass of the body about which the particle moves. Hence, we get,

$$E = -\frac{mc^2}{2EJ^2}(1 - \varepsilon^2)$$

If $\varepsilon < 1$, then energy, E, is negative and the planet remains bound to the centre which in our case is the Sun. $\varepsilon < 1$ signifies that the orbit would be elliptic in nature with the Sun at one of its foci.

Law 2. The radius vector i.e. the line joining the Sun to a given planet sweeps out equal area in equal intervals of time i.e. the areal velocity remains constant.

Suppose the planet moves from P to P' in a small time dt. Hence the area swept would be SPP'. If dt is infinitesimally small then PP' is a straight line $rd\theta$ and SPP' is a triangle. Hence, area of SPP' is given as,

$$dA = \frac{1}{2} r.(rd\theta) = \frac{1}{2} r^2 d\theta$$

Thus, the rate at which this area is swept is given by,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

Under a central force, angular momentum $J = mr^2\dot{\theta}$ is a constant. Hence,

$$r^2\dot{\theta} = \frac{J}{m} = constant$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = constant$$

Law 3. The square of the period of revolution of a planet is proportional to cube of the semi – major axis of the orbit i.e. $T^2 \propto a^3$.

We know that, areal velocity is given by,

$$\frac{dA}{dt} = \frac{J}{2m}$$

On integrating, we get

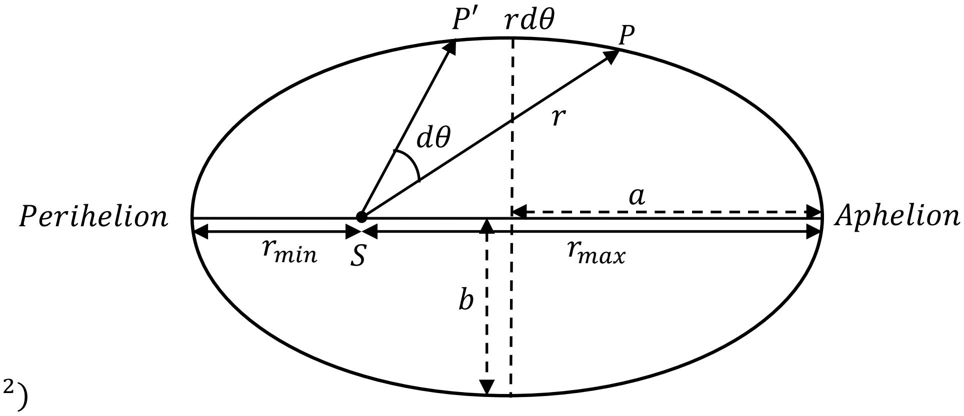
$$\int_0^A dA = \int_0^T \frac{J}{2m} dt$$

$$A = \frac{JT}{2m}$$

Now, area of an ellipse is given by $A = \pi ab$ where a and b are semi – major and semi – minor axis of the ellipse. Hence,

$$\pi ab = \frac{JT}{2m}$$

$$T = \frac{2\pi mab}{J}$$



But, $b = a\sqrt{1-\varepsilon^2}$. Hence,

$$T = \frac{2\pi ma^2\sqrt{1-\varepsilon^2}}{J}$$

On squaring, we get

$$T^2 = \frac{4\pi^2 m^2 a^4 (1-\varepsilon^2)}{J^2}$$

As the origin is taken as the focus S, $2a = r_{max} + r_{min}$.

The polar equation of the orbit under the gravitational force is given as,

$$\frac{1}{r} = \frac{mc}{J^2}\left(1 + \frac{AJ^2}{mc}\cos\theta\right) = \frac{mc}{J^2}(1 + \varepsilon\cos\theta)$$

The maximum and minimum value of $\frac{1}{r}$ will occur at $\theta = 0$ i.e. $\cos\theta = 1$ & $\theta = \pi$ i.e. $\cos\theta = -1$. Hence, we get

$$\left(\frac{1}{r}\right)_{max} = \frac{mc}{J^2}(1+\varepsilon) \quad \& \quad \left(\frac{1}{r}\right)_{min} = \frac{mc}{J^2}(1-\varepsilon)$$

Thus,

$$r_{min} = \frac{J^2}{mc}\frac{1}{1+\varepsilon} \quad \& \quad r_{max} = \frac{J^2}{mc}\frac{1}{1-\varepsilon}$$

Hence,

$$2a = \frac{J^2}{mc}\left[\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right] = \frac{J^2}{mc}\left[\frac{2}{1-\varepsilon^2}\right]$$

$$a = \frac{J^2}{mc}\left[\frac{1}{1-\varepsilon^2}\right]$$

Therefore, the time period of revolution of the planet around the Sun is given by

$$T^2 = \frac{4\pi^2 m^2 a^3}{c}$$

$$T^2 \propto a^3$$

BIBLIOGRAPHY:

1) A Textbook of First Year Physics – S. P. Singh, M. K. Bagde
2) Physics For Degree Students – C. L. Arora, P. S. Hemne
3) Elements of Properties of Matter – D. S. Mathur
4) Concepts of Physics Part 1 – H. C. Verma
5) Fundamentals of Physics – Halliday, Resnick, Walker