MOMENT OF INERTIA

BSc – I (UNIT IV)

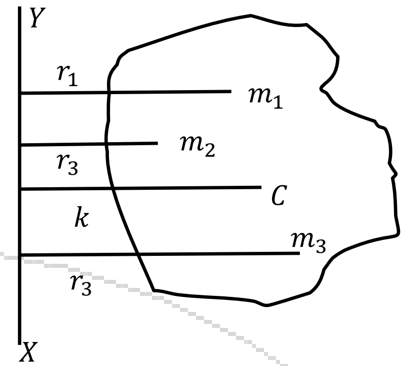
Contents

Moment of Inertia

For translator motion, value of inertia depends only on the mass of the body. The kinetic energy in such motion depends on the mass and linear velocity of the body.

When a body rotates about an axis, the kinetic energy of rotation is determined not only by its mass and angular velocity but also upon the position of the axis about which it rotates and distribution of mass about this axis.

Let us consider a body of mass m rotating about an axis XY with angular velocity ω. All its particles have the same angular velocity but as they are at different distances from the axis of rotation, their linear velocities are different. Hence, we get



$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots$$
$$= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \cdots$$
$$= \frac{1}{2}\left(\sum mr^2\right)\omega^2$$
$$= \frac{1}{2}I\omega^2$$

where, I is the moment of inertia of the body.

Radius of Gyration

If the entire mass of the body is supposed to be concentrated at a point C such that the kinetic energy of rotation is the same as that of the body itself, then the distance of that point from the axis of rotation is called the radius of gyration of the body about that axis.

$$KE = \frac{1}{2}I\omega^2 = \frac{1}{2}Mk^2\omega^2$$
$$\Rightarrow \quad I = Mk^2 = \sum mr^2 = nm\left(\frac{r_1^2 + r_2^2 + \cdots}{n}\right)$$

where, n is the number of particles each of mass m into which the given mass M is divided.

$$\Rightarrow \quad k = \sqrt{\frac{r_1^2 + r_2^2 + \cdots}{n}}$$

The dimension of moment of inertia is $[M^1L^2T^0]$ and its unit is Kgm^2.

The dimension of radius of gyration is $[M^0L^1T^0]$ and its unit is m.

Moment of inertia is scalar because its value about a given axis remains unchanged by reversing its direction of rotation about that axis.

Greater the moment of inertia of a body, greater is the couple required to produce a given angular acceleration.

Theorem of Perpendicular Axes

The moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moment of inertias of the lamina about the two axes at right angles to each other in its own plane intersecting each other at the point where the perpendicular axis passes through it.

$$I = I_X + I_Y$$

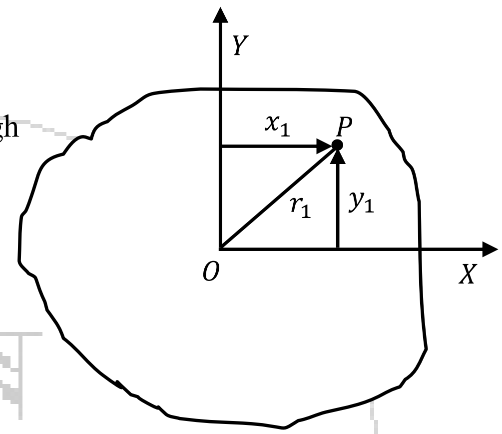
Let OX and OY be two perpendicular axes in the plane of the lamina. Let m_1 be the mass of a particle at point P at distance r_1 from an axis through origin O perpendicular to plane XOY.

Moment of inertia of the particle about X- axis = $m_1 y_1^2$

Moment of inertia of the particle about Y- axis = $m_1 x_1^2$

If we divide the whole lamina into a number of particles of masses $m_1, m_2, m_3, \ldots$ at distances $r_1, r_2, r_3, \ldots$ from the axis. Hence, the moment of inertia about X- axis and Y- axis are given as,

$$I_X = m_1 y_1^2 + m_2 y_2^2 + \cdots = \sum my^2$$
$$I_Y = m_1 x_1^2 + m_2 x_2^2 + \cdots = \sum mx^2$$

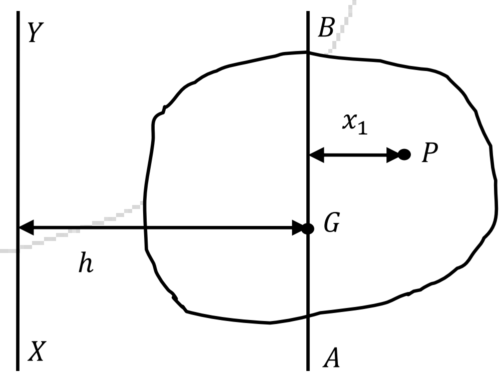


Moment of inertia of the lamina about a perpendicular axis through origin O is given as,

$$\begin{aligned} I &= m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum mr^2 \\ &= m_1(x_1^2 + y_1^2) + m_2(x_2^2 + y_2^2) + \cdots \\ &= m_1 x_1^2 + m_2 x_2^2 + \cdots + m_1 y_1^2 + m_2 y_2^2 + \cdots \\ &= \sum mx^2 + \sum my^2 \\ &= I_Y + I_X \end{aligned}$$

Theorem of Parallel Axes

The moment of inertia of a body about any axis is equal to the sum of its moment of inertias about a parallel axis through its centre of gravity and the product of its mass and the square of the distance between the two axes.



Let XY be an axis in the plane of paper and AB a parallel axis through G, the centre of mass of the body. The perpendicular distance between the two axes is h. Let M be the mass of the body and m_1, the mass of the element at P at a distance x_1 from AB.

Moment of inertia of the element having mass m_1 about axis $XY = m_1(x_1 + h)^2$

$$= m_1 x_1^2 + m_1 h^2 + 2m_1 x_1 h$$

Moment of inertia of the body about axis $XY = I = \sum m_i x_i^2 + \sum m_i h^2 + 2\sum m_i x_i h$

Let I_G be the moment of inertia of the body about AB, an axis through G. Hence, $I_G = \sum m_i x_i^2$.

Hence, moment of inertia of the body is given as

$$I = I_G + Mh^2 + 2h\sum m_i x_i$$

$\sum m_i x_i$ is sum of the moments of all the particles about AB passing through G, the centre of gravity. Since the body is balanced about the centre of mass G, algebraic sum of all the moments about G is zero. Therefore, $\sum m_i x_i = 0$. Hence,

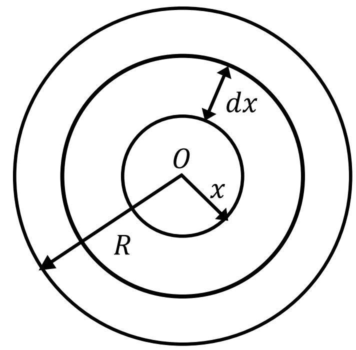
$$I = I_G + Mh^2$$

Moment of Inertia of Geometrical Bodies

Circular Disc

a) About an axis through the centre of the disc perpendicular to its plane

Let us consider a circular disc of radius R. Let us consider an elementary ring of radius x and width dx. The area of the disc is given as $2\pi x dx$. Mass per unit area of the circular disc is $M/_{\pi R^2}$. Hence, mass of the elementary ring is given as,



$$= \frac{M}{\pi R^2} 2\pi x\, dx$$

$$= \frac{2M}{R^2} x\, dx$$

Moment of inertia of the element about an axis through its centre perpendicular to its plane is given as,

$$= \frac{2M}{R^2} x\, dx \,.\, x^2$$

$$= \frac{2M}{R^2} x^3\, dx$$

Hence, the moment of inertia is given as,

$$\boldsymbol{I} = \frac{2M}{R^2} \int_0^R x^3\, dx = \boldsymbol{\frac{1}{2} M R^2}$$

b) About the diameter of the circular disc

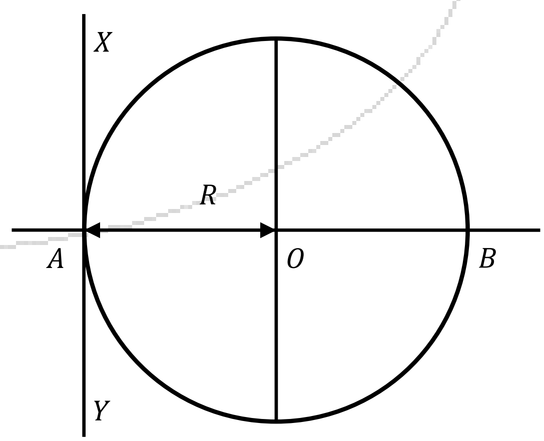
Let us consider AB to be the diameter. Let I_A and I_B be the moment of inertia about the end points of the diameter. Therefore,

$$I = I_A + I_B$$

By the symmetry of the figure, $I_A = I_B$. Hence,

$$I = 2I_A$$

$$\Rightarrow \quad \boldsymbol{I_A} = \frac{I}{2} = \boldsymbol{\frac{1}{4} M R^2}$$



c) About the tangent of the circular disc

Let XY be a tangent at A.

By the theorem of parallel axes,

$$I_T = I_A + Mh^2$$

$$I_T = I_A + M(OA)^2$$

$$I_T = \frac{1}{4} MR^2 + MR^2$$

$$\boldsymbol{I_T} = \boldsymbol{\frac{5}{4} M R^2}$$

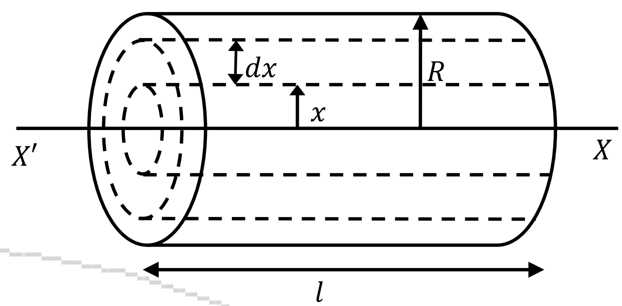
Solid Cylinder

a) About its own axis of symmetry

Let us consider a solid cylinder of mass M, length l and radius R. The volume of the cylinder is given as, $\pi R^2 l$. Hence, the mass density of the cylinder is $M/_{\pi R^2 l}$. Now let us consider a coaxial cylinder of width dx at distance x from the axis of symmetry XX'. Hence, the volume of the coaxial cylinder is $2\pi x l\, dx$. Hence, the mass of the coaxial cylinder is given as,

$$= \frac{M}{\pi R^2 l} 2\pi x l\, dx$$

$$= \frac{2M}{R^2} x\, dx$$



Moment of inertia of the coaxial cylinder is,

$$= \left(\frac{2M}{R^2} x\, dx\right) x^2$$

$$= \left(\frac{2M}{R^2} x^3\, dx\right)$$

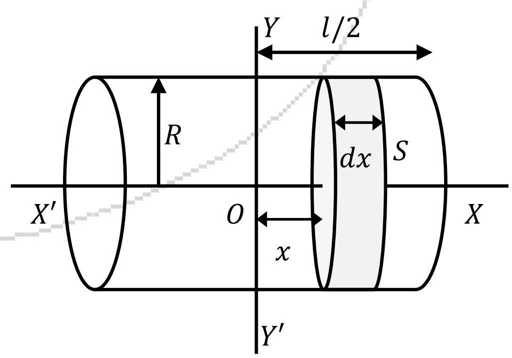
Hence, the moment of inertia is given as,

$$\boldsymbol{I} = \frac{2M}{R^2} \int_0^R x^3\, dx = \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{M R^2}$$

b) About the axis passing through the centre and perpendicular to its own axis of symmetry

Let XX' be the axis of symmetry and YY' be the axis perpendicular to XX'. Let us consider a circular disc S of width dx at a distance x from YY' axis. Mass per unit length of the cylinder is M/l. Hence the mass of the disc is $\frac{M}{l} dx$. Moment of inertia of this disc about the diameter of the rod is,

$$= \left(\frac{M}{l} dx\right) \frac{R^2}{4}$$



Moment of inertia of the disc about YY' axis is given by parallel axes theorem is,

$$= \left(\frac{M}{l} dx\right) \frac{R^2}{4} + \left(\frac{M}{l} dx\right) x^2$$

Hence, the moment of inertia of the cylinder is given as,

$$I = \int_{-l/2}^{l/2} \left(\frac{M}{l} dx\right) \frac{R^2}{4} + \int_{-l/2}^{l/2} \left(\frac{M}{l} dx\right) x^2$$

$$I = \frac{M}{l} \left[\int_{-\frac{l}{2}}^{\frac{l}{2}} \left(\frac{R^2}{4} + x^2\right) dx\right]$$

$$I = \frac{M}{l} \left[\frac{R^2 x}{4} + \frac{x^3}{3}\right]_{-l/2}^{l/2}$$

$$\boldsymbol{I} = \boldsymbol{M} \left[\frac{\boldsymbol{R^2}}{\mathbf{4}} + \frac{\boldsymbol{l^2}}{\mathbf{12}}\right]$$

For a thin rod, $R \approx 0$. Hence moment of inertia is given as,

$$\boldsymbol{I} = \frac{\boldsymbol{M l^2}}{\mathbf{12}}$$

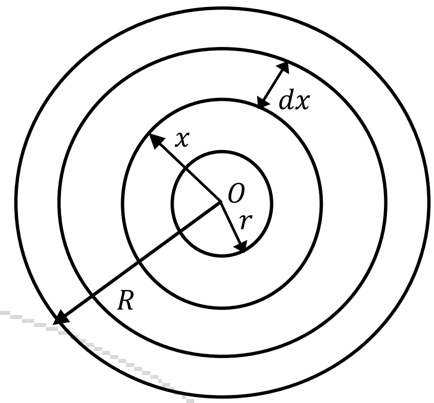
Annular Ring

a) About an axis passing through the origin and perpendicular to its plane

Let us consider a ring having inner radius r and outer radius R having mass M. Area of the face of the ring is $\pi(R^2 - r^2)$. Mass per unit area of the ring is given as, $M/_{\pi(R^2 - r^2)}$. Let us now consider a ring having radius x and $x + dx$. Face area of this ring is $2\pi x\ dx$. Mass of this ring is,

$$= \frac{M}{\pi(R^2 - r^2)} 2\pi x\ dx$$

$$= \frac{2M}{(R^2 - r^2)} x\ dx$$



Moment of inertia of this ring is given as,

$$= \frac{2M}{R^2 - r^2} x\ dx\ .x^2$$

$$= \frac{2M}{R^2 - r^2} x^3\ dx$$

Hence, the moment of inertia is given as,

$$I = \frac{2M}{R^2 - r^2} \int_r^R x^3\ dx = \frac{2M}{R^2 - r^2} \left[\frac{R^4 - r^4}{4}\right]$$

$$\boldsymbol{I = \frac{M}{2}[R^2 + r^2]}$$

b) About its diameter

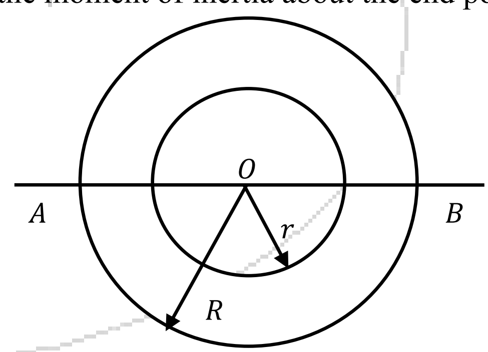
Let us consider AB to be the diameter. Let I_A and I_B be the moment of inertia about the end points of the diameter. Therefore,

$$I = I_A + I_B$$

By the symmetry of the figure, $I_A = I_B$. Hence,

$$I = 2I_A$$

$$\Rightarrow \quad \boldsymbol{I_A} = \frac{I}{2} = \boldsymbol{\frac{1}{4}[R^2 + r^2]}$$



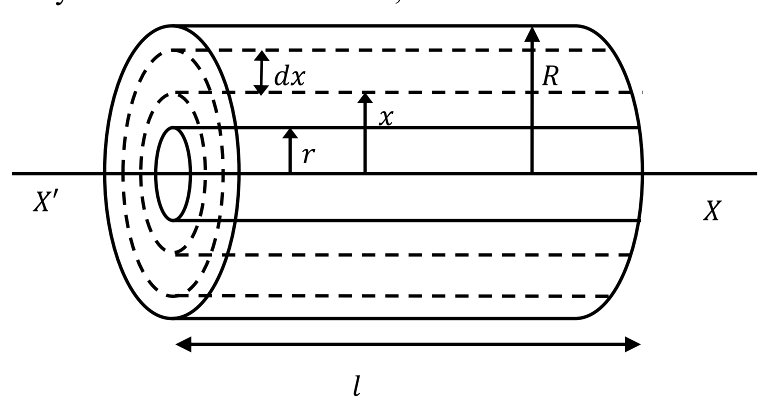
Hollow Cylinder

a) About its own axis of symmetry

Let us consider a hollow cylinder of mass M, length l and inner radius r and outer radius R. The volume of the cylinder is given as, $\pi(R^2 - r^2)l$. Hence, the mass density of the cylinder is $M/_{\pi(R^2 - r^2)l}$. Now let us consider a coaxial cylinder of width dx at distance x from the axis of symmetry. Hence, the volume of the coaxial cylinder is $2\pi xl\ dx$. Hence, the mass of the coaxial cylinder is given as,

$$= \frac{M}{\pi(R^2 - r^2)l} 2\pi xl\ dx$$

$$= \frac{2M}{R^2 - r^2} x\ dx$$



Moment of inertia of the coaxial cylinder is,

$$= \left(\frac{2M}{R^2 - r^2} x\, dx\right) x^2$$

$$= \left(\frac{2M}{R^2 - r^2} x^3\, dx\right)$$

Hence, the moment of inertia is given as,

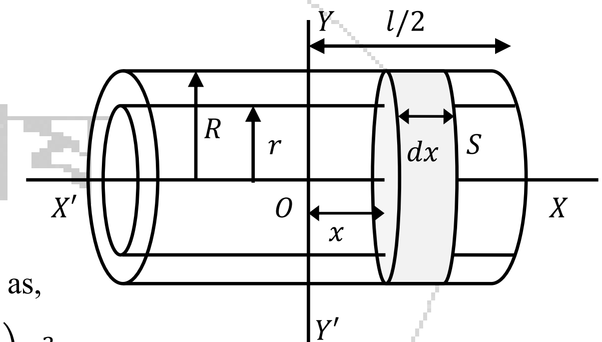
$$I = \frac{2M}{R^2 - r^2} \int_r^R x^3\, dx = \frac{2M}{R^2 - r^2}\left[\frac{R^4 - r^4}{4}\right]$$

$$\boldsymbol{I = \frac{M}{2}[R^2 + r^2]}$$

b) <u>About the axis passing through the centre and perpendicular to its own axis of symmetry</u>

Let XX' be the axis of symmetry and YY' be the axis perpendicular to XX'. Let us consider a circular disc S of width dx at a distance x from YY' axis. Mass per unit length of the cylinder is M/l. Hence the mass of the disc is $\frac{M}{l}dx$. Moment of inertia of this disc about the diameter of the rod is,

$$= \left(\frac{M}{l}dx\right)\frac{R^2 + r^2}{4}$$



Moment of inertia of the disc about YY' axis is given by parallel axes theorem is,

$$= \left(\frac{M}{l}dx\right)\frac{R^2 + r^2}{4} + \left(\frac{M}{l}dx\right)x^2$$

Hence, the moment of inertia of the cylinder is given as,

$$I = \int_{-l/2}^{l/2}\left(\frac{M}{l}dx\right)\frac{R^2 + r^2}{4} + \int_{-l/2}^{l/2}\left(\frac{M}{l}dx\right)x^2$$

$$I = \frac{M}{l}\left[\int_{-\frac{l}{2}}^{\frac{l}{2}}\left(\frac{R^2 + r^2}{4} + x^2\right)dx\right]$$

$$I = \frac{M}{l}\left[\frac{(R^2 + r^2)x}{4} + \frac{x^3}{3}\right]_{-l/2}^{l/2}$$

$$\boldsymbol{I = M\left[\frac{R^2 + r^2}{4} + \frac{l^2}{12}\right]}$$

Solid Sphere

a) <u>About its diameter</u>

Let us consider a solid sphere of radius R and mass M. Consider a thin circular slice of radius, $y = \sqrt{R^2 - x^2}$. The volume of the slice is $\pi(R^2 - x^2)dx$. Let ρ be the mass per unit volume of the sphere. Hence, mass of the slice is given as, $\rho\pi(R^2 - x^2)dx$. Moment of inertia of this slice about a diameter AB is given as,

$$= \frac{1}{2}[\rho\pi(R^2 - x^2)dx]y^2$$

$$= \frac{1}{2}[\rho\pi(R^2 - x^2)^2 dx]$$

Hence, the moment of inertia of the disc about its diameter is,

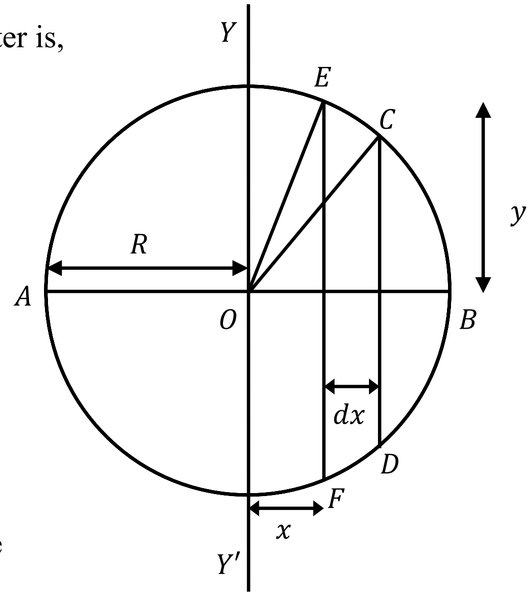
$$I = \int_{-R}^{R} \frac{1}{2}\rho\pi(R^2 - x^2)^2 dx$$

$$I = 2 \times \frac{1}{2}\rho\pi \int_{0}^{R} (R^2 - x^2)^2 dx$$

$$I = \rho\pi \int_{0}^{R} (R^4 - 2R^2x^2 + x^4) dx$$

$$I = \rho\pi \left[R^4x - 2R^2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^R$$

$$I = \frac{8\rho\pi}{15}R^5$$



But mass of the sphere is $M = \frac{4}{3}\pi R^3\rho$. Hence, the above expression can be written as,

$$I = 2\left(\frac{4}{3}\pi R^3\rho\right)\left(\frac{R^2}{5}\right)$$

$$\boldsymbol{I = \frac{2}{5}MR^2}$$

b) <u>About a tangent</u>

Let XY be a tangent at A.

By the theorem of parallel axes,

$$I_T = I_A + Mh^2$$

$$I_T = I_A + M(OA)^2$$

$$I_T = \frac{2}{5}MR^2 + MR^2$$

$$\boldsymbol{I_T = \frac{7}{5}MR^2}$$

<u>Thin Spherical Shell</u>

<u>About its diameter</u>

Let us consider a thin spherical shell of radius R and mass M. The mass per unit area of the shell is given as, $M/_{4\pi R^2}$. Let us consider a thin element of shell bounded by two parallel planes EF & CD at x and $x + dx$. Let the radius of the shell be y and its thickness be EC.

The area of the thin element $=$ $circumference \times width$

$$= (2\pi y)(Rd\theta)$$

By the geometry of the figure, $y = R\cos\theta$ and $x = R\sin\theta$.

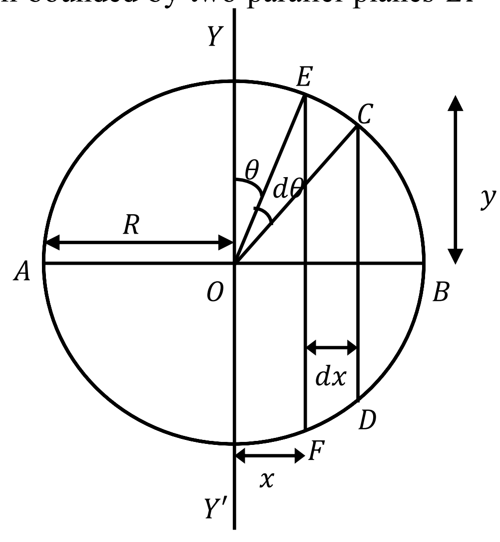
Differentiating we get,

$$dx = R\cos\theta = yd\theta$$

And, $y = \sqrt{R^2 - x^2}$

Area of the thin element $= 2\pi R dx$

Mass of the thin element $= \left(\frac{M}{4\pi R^2}\right)2\pi R dx = \frac{M}{2R}dx$



Hence, the moment of inertia about the diameter,

$$= \left(\frac{M}{2R}dx\right)y^2$$

$$= \frac{M}{2R}(R^2 - x^2)dx$$

The moment of inertia about the diameter of the spherical shell is given as,

$$I = \int_{-R}^{R} \frac{M}{2R}(R^2 - x^2)dx$$

$$I = \frac{M}{2R}\left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R}$$

$$\boldsymbol{I = \frac{2}{3}MR^2}$$

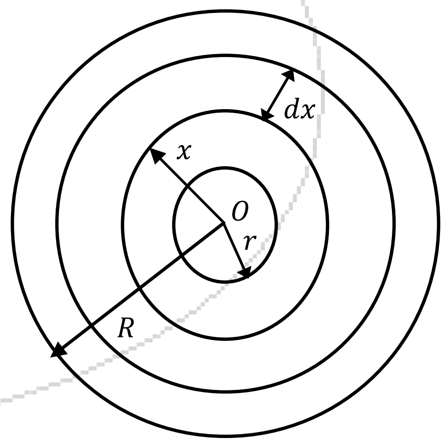
Thick Spherical Shell

About its diameter

Let us consider a thick spherical shell of inner radius r, outer radius R and mass M. The total volume of the shell is $\frac{4}{3}\pi(R^3 - r^3)$. Hence, the mass per unit volume of the shell is $3M/4\pi(R^3 - r^3)$. Let us consider a concentric shell of thickness dx at a distance x from the centre. Hence, the mass of this concentric shell is,

$$= \left(\frac{3M}{4\pi(R^3 - r^3)}\right)4\pi x^2\, dx$$

$$= \left(\frac{3M}{R^3 - r^3}\right)x^2\, dx$$



Moment of inertia of this thin shell about the diameter is given as,

$$= \frac{2}{3}\left[\left(\frac{3M}{R^3 - r^3}\right)x^2\, dx\right]x^2$$

$$= \left(\frac{2M}{R^3 - r^3}\right)x^4\, dx$$

Moment of inertia of the hollow sphere about its diameter is,

$$I = \int_{r}^{R}\left(\frac{2M}{R^3 - r^3}\right)x^4\, dx$$

$$I = \frac{2M}{R^3 - r^3}\left[\frac{x^5}{5}\right]_{r}^{R}$$

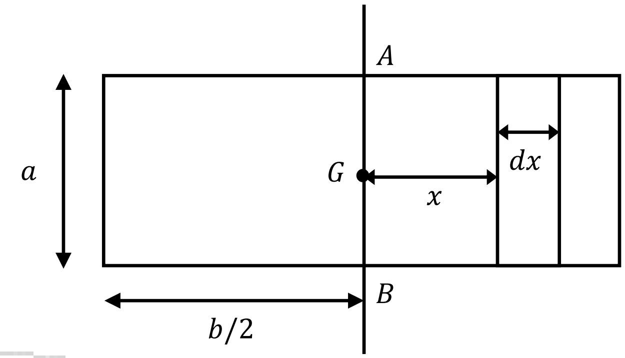
$$\boldsymbol{I = \frac{2M}{5}\left[\frac{R^5 - r^5}{R^3 - r^3}\right]}$$

Rectangular Lamina

About an axis perpendicular to its plane and passing through its centre of gravity

Let us consider a rectangular lamina of mass M having length a and breadth b. The surface area of the lamina is ab. Hence, the mass per unit area of the lamina is $M/_{ab}$. Let us consider a thin strip of width dx at a distance x from the centre of gravity G. The area of the strip is ,

$$= a\,dx$$



The mass of the strip is given by,

$$= \left(\frac{M}{ab}\right) a\,dx$$

$$= \frac{M}{b} dx$$

The moment of inertia of the strip about an axis AB parallel to side a is given by,

$$= \left(\frac{M}{b} dx\right) x^2$$

The moment of inertia of the lamina is given as,

$$I = \int_{-b/2}^{b/2} \frac{M}{b} x^2 dx$$

$$I = \frac{M}{b}\left[\frac{x^3}{3}\right]_{-b/2}^{b/2}$$

$$\boldsymbol{I_Y = \frac{Mb^2}{12}}$$

The moment of inertia of the lamina about an axis parallel to side b is given by,

$$\boldsymbol{I_X = \frac{Ma^2}{12}}$$

Applying perpendicular axes theorem, we find the moment of inertia of the lamina about an axis perpendicular to its plane and passing through its centre of gravity as,

$$I = I_X + I_Y$$

$$\boldsymbol{I = \frac{M}{12}(a^2 + b^2)}$$

Rectangular Bar

a) About an axis perpendicular to its plane and passing through its centre of gravity

Let us consider a bar having length l and breadth b having mass M. Let us consider the bar to be made up of laminas placed one above the other. Hence the moment of inertia of the bar about an axis perpendicular to its plane and passing through its centre of gravity as,

$$I_Y = \sum\left(m\frac{l^2 + b^2}{12}\right) = \sum m\left(\frac{l^2 + b^2}{12}\right)$$

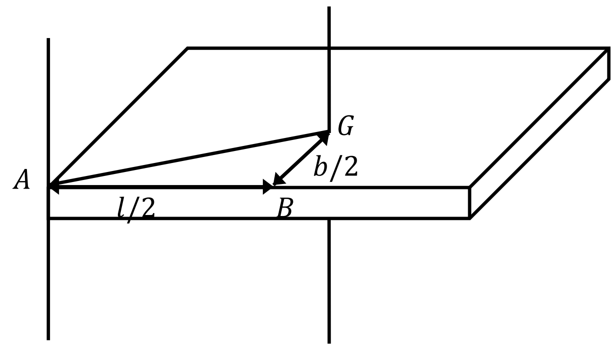
$$\boldsymbol{I_Y = M\left(\frac{l^2 + b^2}{12}\right)}$$

b) About an axis perpendicular to its length and passing through one of its edges

The moment of inertia of the bar through its centre of gravity G is I_Y. Let us find the moment of inertia at the edge A. By the geometry of the figure,

$$AG^2 = AB^2 + BG^2 = \left(\frac{l}{2}\right)^2 + \left(\frac{b}{2}\right)^2$$

$$AG^2 = \frac{l^2 + b^2}{4}$$



By using parallel axes theorem we can find the moment of inertia of the bar about an axis perpendicular to its length and passing through one of its edges as,

$$I = I_Y + M(AG^2)$$

$$I = M\left(\frac{l^2 + b^2}{12}\right) + M\left(\frac{l^2 + b^2}{4}\right)$$

$$\boldsymbol{I = M\left(\frac{l^2 + b^2}{3}\right)}$$

For a square bar, $l = b$. Hence, the moment of inertia is given as,

$$\boldsymbol{I = \frac{2}{3}Mb^2}$$

Solid Cone

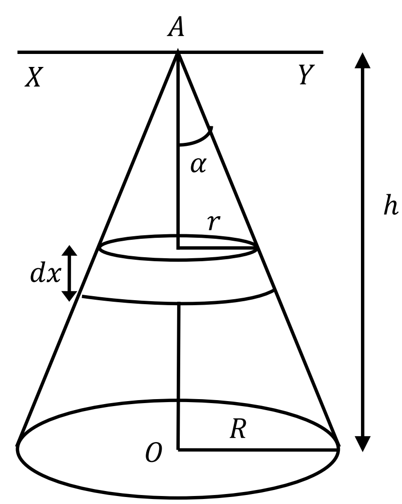
a) About its vertical axis

Let us consider a solid cone having mass M, base radius R and height h. The mass per unit volume of the cone is given as, $\rho = {}^{3M}/_{\pi R^2 h}$. Let α be the semi vertical angle of the cone. Let us consider a small circular disc of radius r at a distance x from the vertex. Let the disc have a thickness dx. Hence, volume of the disc is,

$$= \pi r^2 dx$$

According to the geometry of the figure we have, $\tan\alpha = {}^{R}/_{h}$ and $r = {}^{xR}/_{h}$

Hence, mass of the disc is given by,



$$= \left(\frac{3M}{\pi R^2 h}\right)\pi r^2 dx$$

$$= \frac{3Mr^2}{R^2 h}dx$$

Hence, moment of inertia of the disc about the vertical axis perpendicular to its plane is given as,

$$= \frac{1}{2}\left(\frac{3Mr^2}{R^2 h}dx\right)r^2$$

$$= \frac{3M}{2R^2 h}r^4 dx$$

$$= \frac{3MR^2}{2h^5}x^4 dx$$

Hence, the moment of inertia of the solid cone about its vertical axis is given by,

$$I = \int_0^h \frac{3MR^2}{2h^5} x^4 dx$$

$$I = \frac{3MR^2}{2h^5}\left[\frac{x^5}{5}\right]_0^h$$

$$\boldsymbol{I} = \frac{3MR^2}{2h^5}\frac{h^5}{5} = \boldsymbol{\frac{3}{10}MR^2}$$

b) <u>About an axis through its vertex and parallel to the base</u>

Let us consider an axis XY through the vertex parallel to the base. Let us consider a small circular disc of radius r at a distance x from the vertex. Let the disc have a thickness dx. Hence, the moment of inertia of this disc about its diameter is given as,

$$= \frac{1}{4}\left(\frac{3Mr^2}{R^2h}dx\right)r^2$$

By parallel axis theorem, the moment of inertia of the disc about the axis parallel through the vertex and parallel to the diameter of the disc is given as,

$$= \frac{1}{4}\left(\frac{3Mr^2}{R^2h}dx\right)r^2 + \left(\frac{3Mr^2}{R^2h}dx\right)x^2$$

$$= \frac{3M}{4R^2h}r^4dx + \left(\frac{3Mr^2}{R^2h}dx\right)x^2$$

$$= \frac{3MR^2}{4h^5}x^4dx + \frac{3M}{h^3}x^4dx$$

Hence, the moment of inertia of the solid cone about the axis parallel through the vertex and parallel to the base is given as,

$$I = \int_0^h \frac{3MR^2}{4h^5}x^4dx + \int_0^h \frac{3M}{h^3}x^4dx$$

$$I = \frac{3MR^2}{4h^5}\left[\frac{x^5}{5}\right]_0^h + \frac{3M}{h^3}\left[\frac{x^5}{5}\right]_0^h$$

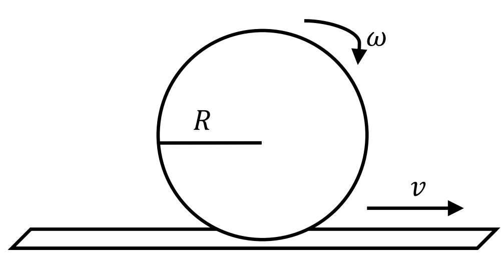
$$\boldsymbol{I} = \boldsymbol{\frac{3MR^2}{20} + \frac{3Mh^2}{5}}$$

Kinetic Energy of a Body Rolling on a Horizontal Plane

Let us consider a body of mass M having radius R rolling on a horizontal plane. Let v be the linear velocity and ω be the angular velocity of the body. The linear and rotational kinetic energy are given as,

Linear kinetic energy $= \frac{1}{2}Mv^2$

Rotational kinetic energy $= \frac{1}{2}I\omega^2$



Hence, the total kinetic energy is given as,

$$\text{Total energy} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}Mk^2\omega^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}Mk^2\frac{v^2}{R^2}$$
$$= \boldsymbol{\frac{1}{2}Mv^2\left(1 + \frac{k^2}{R^2}\right)}$$

<u>Special Cases</u>

a) <u>Spherical Ball</u>

For a spherical ball moment of inertia about its diameter is $\frac{2}{5}MR^2$. Hence,

$$I = Mk^2 = \frac{2}{5}MR^2$$
$$\Rightarrow k^2 = \frac{2}{5}R^2$$
$$\Rightarrow \frac{k^2}{R^2} = \frac{2}{5}$$

Therefore, the total kinetic energy of the rolling spherical ball is given by,

$$\text{Total Kinetic Energy} = \frac{1}{2}Mv^2\left(1 + \frac{k^2}{R^2}\right) = \frac{1}{2}Mv^2\left(1 + \frac{2}{5}\right)$$
$$\textbf{Total Kinetic Energy} = \boldsymbol{\frac{7}{10}Mv^2}$$

b) <u>Circular Disc</u>

For a circular disc moment of inertia about an axis through its centre and perpendicular to its plane is $\frac{1}{2}MR^2$. Hence,

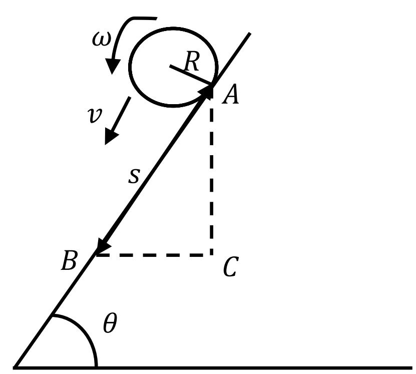
$$I = Mk^2 = \frac{1}{2}MR^2$$
$$\Rightarrow k^2 = \frac{1}{2}R^2$$
$$\Rightarrow \frac{k^2}{R^2} = \frac{1}{2}$$

Therefore, the total kinetic energy of the rolling circular disc is given by,

$$\text{Total Kinetic Energy} = \frac{1}{2}Mv^2\left(1 + \frac{k^2}{R^2}\right) = \frac{1}{2}Mv^2\left(1 + \frac{1}{2}\right)$$
$$\textbf{Total Kinetic Energy} = \boldsymbol{\frac{3}{4}Mv^2}$$

<u>Acceleration of a Body Rolling Down an Inclined Plane</u>

Let us consider a body of mass M having radius R rolling on a inclined plane. Let v be the linear velocity and ω be the angular velocity of the body. In one revolution, it will move a distance of $2\pi R$. The total kinetic energy of the moving body is, $\frac{1}{2}Mv^2\left(1 + \frac{k^2}{R^2}\right)$. In moving a distance $AB = s$, the vertical distance travelled is $s\sin\theta$. Hence the change in potential energy is $Mgs\sin\theta$.
Since, there is no slipping, no energy is dissipated. Hence,



$$Potential\ Energy = Total\ Kinetic\ Energy$$
$$\Rightarrow\ Mgs\sin\theta = \frac{1}{2}Mv^2\left(1 + \frac{k^2}{R^2}\right)$$
$$\Rightarrow\ gs\sin\theta = \frac{1}{2}v^2\left(1 + \frac{k^2}{R^2}\right)$$

Differentiating the above equation with respect to t, we get

$$\Rightarrow\; g\sin\theta\frac{ds}{dt} = v\frac{dv}{dt}\left(1+\frac{k^2}{R^2}\right)$$

Now, $\frac{ds}{dt}$ is the velocity v of the body and $\frac{dv}{dt}$ is the acceleration a of the body. Hence,

$$\Rightarrow\; gv\sin\theta = va\left(1+\frac{k^2}{R^2}\right)$$

$$\Rightarrow\; \boldsymbol{a = \frac{g\sin\theta}{1+k^2/R^2}}$$

<u>Special Cases</u>

a) <u>Solid Cylinder</u>

For a solid cylinder, moment of inertia about an axis through its centre is $\frac{1}{2}MR^2$. Hence,

$$I = Mk^2 = \frac{1}{2}MR^2$$

$$\Rightarrow k^2 = \frac{1}{2}R^2$$

$$\Rightarrow \frac{k^2}{R^2} = \frac{1}{2}$$

Therefore, the total acceleration of the rolling solid cylinder is given by,

Acceleration, $a = \frac{g\sin\theta}{1+k^2/R^2} = \frac{g\sin\theta}{1+1/2}$

Acceleration, $\boldsymbol{a = \frac{2}{3}g\sin\theta}$

b) <u>Solid Sphere</u>

For a solid sphere, moment of inertia about its diameter is $\frac{2}{5}MR^2$. Hence,

$$I = Mk^2 = \frac{2}{5}MR^2$$

$$\Rightarrow k^2 = \frac{2}{5}R^2$$

$$\Rightarrow \frac{k^2}{R^2} = \frac{2}{5}$$

Therefore, the total acceleration of the rolling solid sphere is given by,

Acceleration, $a = \frac{g\sin\theta}{1+k^2/R^2} = \frac{g\sin\theta}{1+2/5}$

Acceleration, $\boldsymbol{a = \frac{5}{7}g\sin\theta}$

c) <u>Hollow Sphere</u>

For a hollow sphere, moment of inertia about its diameter is $\frac{2}{3}MR^2$. Hence,

$$I = Mk^2 = \frac{2}{3}MR^2$$

$$\Rightarrow k^2 = \frac{2}{3}R^2$$

$$\Rightarrow \frac{k^2}{R^2} = \frac{2}{3}$$

Therefore, the total acceleration of the rolling hollow sphere is given by,

Acceleration, $a = \frac{g\sin\theta}{1+k^2/R^2} = \frac{g\sin\theta}{1+2/3}$

Acceleration, $\boldsymbol{a = \frac{3}{5}g\sin\theta}$

Products of Moment of Inertia

Let us consider a rigid body of mass M composing of n particles. For any i^{th} particle, angular momentum is given by,

$$\vec{J_i} = \vec{r_i} \times \vec{p_i}$$

$$\Rightarrow \vec{J_i} = m_i(\vec{r_i} \times \vec{v_i})$$

But, $\vec{v_i} = \vec{\omega} \times \vec{r_i}$. Hence,

$$\Rightarrow \vec{J_i} = m_i[\vec{r_i} \times (\vec{\omega} \times \vec{r_i})]$$

Hence, total angular momentum of the rigid body is given by,

$$\vec{J} = \sum_i m_i[\vec{r_i} \times (\vec{\omega} \times \vec{r_i})]$$

But, $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \,.\, \vec{C}) - \vec{C}(\vec{A} \,.\, \vec{B})$. Hence, the above equation becomes,

$$\vec{J} = \sum_i m_i[\omega(\vec{r_i} \,.\, \vec{r_i}) - \vec{r_i}(\vec{r_i} \,.\, \vec{\omega})\,]$$

$$\Rightarrow \vec{J} = \sum_i m_i r_i^2 \omega - \sum_i m_i \vec{r_i}(\vec{r_i} \,.\, \vec{\omega})$$

The x - component of the above expression is,

$$J_x = \omega_x \sum_i m_i r_i^2 - \omega_x \sum_i m_i x_i^2 - \omega_y \sum_i m_i x_i\, y_i - \omega_z \sum_i m_i x_i\, z_i$$

$$J_x = \omega_x \sum_i m_i(r_i^2 - x_i^2) - \omega_y \sum_i m_i x_i\, y_i - \omega_z \sum_i m_i x_i\, z_i$$

Similarly equations for J_y and J_z can be written as,

$$J_y = \omega_y \sum_i m_i(r_i^2 - y_i^2) - \omega_x \sum_i m_i y_i\, x_i - \omega_z \sum_i m_i y_i\, z_i$$

$$J_z = \omega_z \sum_i m_i(r_i^2 - z_i^2) - \omega_x \sum_i m_i z_i\, x_i - \omega_y \sum_i m_i z_i\, y_i$$

Since, $r_i^2 = x_i^2 + y_i^2 + z_i^2$, we get

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2) = \sum_i m_i(y_i^2 + z_i^2) \qquad I_{xy} = -\sum_i m_i x_i\, y_i \qquad I_{xz} = -\sum_i m_i x_i\, z_i$$

$$I_{yy} = \sum_i m_i(r_i^2 - y_i^2) = \sum_i m_i(x_i^2 + z_i^2) \qquad I_{yx} = -\sum_i m_i y_i\, x_i \qquad I_{yz} = -\sum_i m_i y_i\, z_i$$

$$I_{zz} = \sum_i m_i(r_i^2 - z_i^2) = \sum_i m_i(x_i^2 + y_i^2) \qquad I_{zx} = -\sum_i m_i z_i\, x_i \qquad I_{zy} = -\sum_i m_i z_i\, y_i$$

Hence, the equations for angular momentum can be written as,

$$J_x = I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z$$

$$J_y = I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z$$

$$J_z = I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z$$

This can be written as,

$$J_j = \sum_k I_{jk}\,\omega_k$$

$$\vec{J} = \prod .\, \omega$$

where,

$$\prod = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

The diagonal elements of the matrix are called the **Coefficient of Moment of Inertia.**
The other elements of the matrix are called the **Products of Moment of Inertia.**

Principal Moments

Principal moments can be solved by finding the eigen values of the above matrix,

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$$

One can find a set of cartesian axes for which the inertia tensor will be a diagonal tensor. The axes are called as the **Principal axes** and the corresponding diagonal components are called **Principal moments of inertia**.

The angular momentum is given by,

$$\vec{J_j} = \sum_k I_{jk}\, \omega_k$$

We know that, torque applied to a body is the rate change of angular momentum. Hence,

$$\vec{\tau_j} = \frac{d\vec{J_j}}{dt}$$

Now we know that,

$$\frac{d\vec{r}}{dt} = \frac{\delta \vec{r}}{\delta t} + \vec{\omega} \times \vec{r}$$

Hence, $\frac{d}{dt} = \frac{\delta}{\delta t} + \vec{\omega} \times$ can be treated as an operator. Hence, we get

$$\vec{\tau_j} = \frac{d\vec{J_j}}{dt} = \frac{\delta \vec{J_i}}{\delta t} + \vec{\omega} \times \vec{J_j}$$

If we select the axes of rotation in the body where all the products of moment of inertia vanish, i.e.

$$I_{ij} = 0 \qquad for\ i \neq j$$

These axes where the products of moment of inertia vanish are called the **Principal Axes**. Hence,

$$J_1 = I_{11}\omega_1 = I_1\omega_1$$
$$J_2 = I_{22}\omega_2 = I_2\omega_2$$
$$J_3 = I_{33}\omega_3 = I_3\omega_3$$

where, I_1, I_2 and I_3 are called the principal moments of inertia.

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