Centre of mass Collisions

&

Conservation laws

BSc – I (Unit IV)

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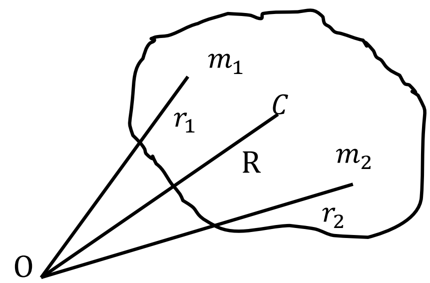
Centre of Mass

Suppose n particles constitute the mass M of a rigid body. Let the masses of the n particles be $m_1, m_2, m_3, \dots, m_i, \dots, m_n$ whose position vectors from the origin O are $\vec{r_1}, \vec{r_2}, \vec{r_3}, \dots, \vec{r_i}, \dots, \vec{r_n}$. Then the position vector of the centre of mass, $\vec{R}$ is defined as,



$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \dots + m_i\vec{r_i} + \dots + m_n\vec{r_n}}{m_1 + m_2 + m_3 + \dots + m_i + \dots + m_n}$$

$$\boldsymbol{\vec{R}} = \frac{\sum_{i=1}^{n} m_i\vec{r_i}}{\sum_{i=1}^{n} m_i} = \frac{\boldsymbol{\sum_{i=1}^{n} m_i\vec{r_i}}}{\boldsymbol{M}}$$

If the centre of mass coincides with the origin of the system then

$$\vec{R} = 0 \ \Rightarrow \sum m_i\vec{r_i} = 0$$

The centre of mass is defined as a point in space such that the vector sum of the moments of the mass points around it is zero.

Velocity of Centre of Mass

We can express the above expression as,

$$M\vec{R} = m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \dots + m_i\vec{r_i} + \dots + m_n\vec{r_n}$$

Differentiating with respect to time t, we get

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r_1}}{dt} + m_2\frac{d\vec{r_2}}{dt} + m_3\frac{d\vec{r_3}}{dt} + \dots + m_i\frac{d\vec{r_i}}{dt} + \dots + m_n\frac{d\vec{r_n}}{dt}$$

$$M\overrightarrow{V_{cm}} = m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \dots + m_i\vec{v_i} + \dots + m_n\vec{v_n}$$

$$\overrightarrow{V_{cm}} = \frac{m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \dots + m_i\vec{v_i} + \dots + m_n\vec{v_n}}{M}$$

$$\boldsymbol{\overrightarrow{V_{cm}}} = \frac{\boldsymbol{\sum_{i=1}^{n} m_i\vec{v_i}}}{\boldsymbol{M}}$$

$\overrightarrow{V_{cm}}$ is called the velocity of the centre of mass. The momentum $\vec{P}$ of the centre of mass is given as $\overrightarrow{P_{cm}} = M\overrightarrow{V_{cm}} = \sum m_i\vec{v_i}$. If no external force is applied the momentum is conserved and hence the velocity remains constant.

Acceleration of Centre of Mass

We have the above expression as,

$$M\overrightarrow{V_{cm}} = m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \dots + m_i\vec{v_i} + \dots + m_n\vec{v_n}$$

Differentiating with respect to time t, we get

$$M\frac{d\overrightarrow{V_{cm}}}{dt} = m_1\frac{d\vec{v_1}}{dt} + m_2\frac{d\vec{v_2}}{dt} + m_3\frac{d\vec{v_3}}{dt} + \dots + m_i\frac{d\vec{v_i}}{dt} + \dots + m_n\frac{d\vec{v_n}}{dt}$$

$$M\overrightarrow{A_{cm}} = m_1\vec{a_1} + m_2\vec{a_2} + m_3\vec{a_3} + \dots + m_i\vec{a_i} + \dots + m_n\vec{a_n}$$

$$\overrightarrow{A_{cm}} = \frac{m_1\vec{a_1} + m_2\vec{a_2} + m_3\vec{a_3} + \dots + m_i\vec{a_i} + \dots + m_n\vec{a_n}}{M}$$

$$\boldsymbol{\overrightarrow{A_{cm}}} = \frac{\boldsymbol{\sum_{i=1}^{n} m_i\vec{a_i}}}{\boldsymbol{M}}$$

$\overrightarrow{A_{cm}}$ is called the acceleration of the centre of mass. Hence, we can write the vector sum of all external forces on the individual particles to be,

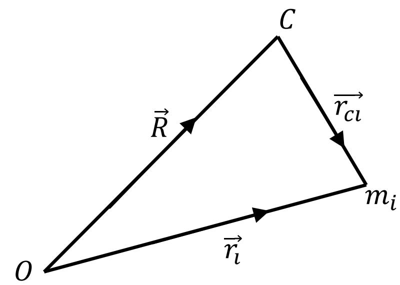
$$M\overrightarrow{A_{cm}} = \sum \vec{F_i}$$

If no external force is applied, the acceleration of the centre of mass is zero.

Linear Momentum of the Particles

Let us consider a mass m_i having the position vector $\vec{r_i}$. Let the centre of mass C have the position vector $\vec{R}$. The position vector between the centre of mass and the mass m_i be $\vec{r_{ci}}$.
Hence, we have

$$\vec{R} + \vec{r_{ci}} = \vec{r_i}$$



We also have,

$$M\vec{R} = \sum m_i \vec{r_i} = \sum m_i(\vec{R} + \vec{r_{ci}}) = M\vec{R} + \sum m_i \vec{r_{ci}}$$

$$\Rightarrow \sum m_i \vec{r_{ci}} = 0 \Rightarrow \sum m_i \frac{d\vec{r_{ci}}}{dt} = 0$$

$$\Rightarrow \sum m_i \vec{v_i} = 0$$

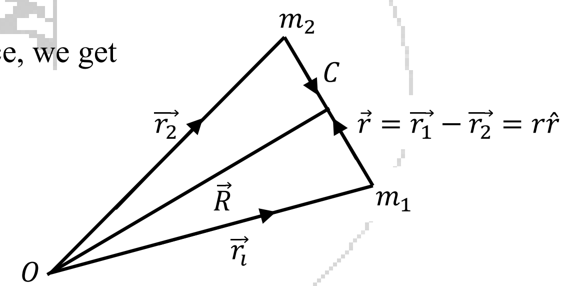
Hence, the total linear momentum about the centre of mass is zero.

Reduced Mass

Let us consider two masses m_1 and m_2 separated by a distance $\vec{r}$. Let their position vectors be respectively $\vec{r_1}$ and $\vec{r_2}$. Let the centre of mass C have the position vector $\vec{R}$. It is given as,

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$

If no external force is applied then the velocity of the centre of mass, $\overrightarrow{V_{cm}}$, is constant.
The force on m_1 due to m_2 and on m_2 due to m_1 are both directed towards the centre of mass. Hence the forces are central forces.
Let the force on m_1 be $F(r)\hat{r}$ and that on m_2 be $-F(r)\hat{r}$. Hence, we get



$$m_1 \frac{d^2\vec{r_1}}{dt^2} = F(r)\hat{r}$$

$$m_2 \frac{d^2\vec{r_2}}{dt^2} = -F(r)\hat{r}$$

On subtracting second equation from the first, we get

$$\frac{d^2\vec{r_1}}{dt^2} - \frac{d^2\vec{r_2}}{dt^2} = F(r)\hat{r}\left[\frac{1}{m_1} + \frac{1}{m_2}\right]$$

$$\frac{d^2}{dt^2}(\vec{r_1} - \vec{r_2}) = F(r)\hat{r}\left[\frac{1}{m_1} + \frac{1}{m_2}\right]$$

But, $\vec{r_1} - \vec{r_2} = \vec{r}$ and let us define the reduced mass as $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$. Hence we get

$$\frac{d^2\vec{r}}{dt^2} = F(r)\hat{r}\frac{1}{\mu}$$

$$\Rightarrow \mu\ddot{\vec{r}} = F(r)\hat{r}$$

If $\vec{F}(r)$ is the gravitational force, then $\vec{F}(r) = -\frac{Gm_1m_2}{r^2}\hat{r}$. Hence, we get

$$\mu\ddot{\vec{r}} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

$$\frac{m_1m_2}{m_1 + m_2}\ddot{\vec{r}} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

Let, $m_1 + m_2 = M$. Hence

$$\ddot{\vec{r}} = -\frac{GM}{r^2}\hat{r}$$

In the centre of mass frame, total linear momentum is conserved. Hence, the velocity of centre of mass remains constant. In the centre of mass frame, $\overrightarrow{V_{cm}} = 0$. Therefore, the two bodies move around the centre of mass in such a way that $\vec{r}$ remains constant. Hence to an observer in the centre of mass frame, heavier mass seems to describe a smaller circle and the lighter a larger circle. Hence,

$$\frac{r_2}{r_1} = -\frac{m_1}{m_2}$$

Collision

When two bodies are approaching each other, a force comes into play between them for a finite time and brings about a measurable change in their velocities, momenta and energy according to the respective laws of conservation, a collision is said to have taken place.

Scattering: If the nature of particles does not change after collision, it is termed as scattering.

Elastic Scattering: If in a scattering, sum of kinetic energies is conserved then it is termed as elastic scattering.

Inelastic Scattering: If in a scattering, sum of kinetic energies is not conserved then it is termed as inelastic scattering.

Reaction: If the final products are different from the initial reactants and the sum of kinetic energies is not conserved, then the collision is said to be a reaction.

Exoergic Reaction: If loss of energy occurs in a reaction then it is called as exoergic reaction.

Endoergic Reaction: If energy adds up in a reaction then it is called as endoergic reaction.

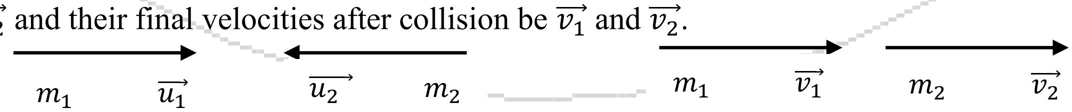
Laboratory Frame: If the origin of the reference system is a point rigidly fixed to the laboratory, it is known as the laboratory frame.

Centre of Mass Frame: If the origin of the reference system is a point rigidly fixed to the centre of mass of a system of particles on which no external force is acting, it is known as the centre of mass frame of reference.

Perfectly Elastic Collision

Laboratory Frame (One – Dimension)

Let us consider two particles having mass m_1 and m_2. Let their initial velocities before the collision be $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ and their final velocities after collision be $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.



Before Collision **After Collision**

Hence, by principal of conservation of linear momentum, we get

$$m_1\overrightarrow{u_1} + m_2\overrightarrow{u_2} = m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2} \qquad (1)$$

By principal of conservation of energy, we get

$$\frac{1}{2}m_1\overrightarrow{u_1^2} + \frac{1}{2}m_2\overrightarrow{u_2^2} = \frac{1}{2}m_1\overrightarrow{v_1^2} + \frac{1}{2}m_2\overrightarrow{v_2^2} \qquad (2)$$

$$(1) \Rightarrow \quad m_1(\overrightarrow{u_1} - \overrightarrow{v_1}) = m_2(\overrightarrow{v_2} - \overrightarrow{u_2}) \qquad (3)$$

$$(2) \Rightarrow \quad m_1\left(\overrightarrow{u_1^2} - \overrightarrow{v_1^2}\right) = m_2(\overrightarrow{v_2^2} - \overrightarrow{u_2^2}) \qquad (4)$$

On dividing (4) by (3), we get,

$$\overrightarrow{u_1} + \overrightarrow{v_1} = \overrightarrow{v_2} + \overrightarrow{u_2}$$

$$\overrightarrow{u_1} - \overrightarrow{u_2} = -(\overrightarrow{v_1} - \overrightarrow{v_2})$$

Hence, we see that the relative velocity is equal. Hence, the velocity after collision is given as,

$$\vec{v_1} = \vec{v_2} + \vec{u_2} - \vec{u_1}$$
$$\vec{v_2} = \vec{v_1} + \vec{u_1} - \vec{u_2}$$

Hence,

(3) $\Rightarrow$
$$m_1(\vec{u_1} - \vec{v_1}) = m_2(\vec{v_1} + \vec{u_1} - \vec{u_2} - \vec{u_2})$$
$$\vec{v_1}(m_1 + m_2) = \vec{u_1}(m_1 - m_2) + 2m_2\vec{u_2}$$
$$\vec{v_1} = \frac{m_1 - m_2}{m_1 + m_2}\vec{u_1} + \frac{2m_2}{m_1 + m_2}\vec{u_2}$$

(3) $\Rightarrow$
$$m_1(\vec{u_1} - \vec{v_2} - \vec{u_2} + \vec{u_1}) = m_2(\vec{v_2} - \vec{u_2})$$
$$\vec{v_2}(m_1 + m_2) = \vec{u_2}(m_2 - m_1) + 2m_1\vec{u_1}$$
$$\vec{v_2} = \frac{m_2 - m_1}{m_1 + m_2}\vec{u_2} + \frac{2m_1}{m_1 + m_2}\vec{u_1}$$

Cases:

i) Let one particle be initially at rest. ($\vec{u_2} = 0$). Hence, the final velocities are given as,
$$\vec{v_1} = \frac{m_1 - m_2}{m_1 + m_2}\vec{u_1}$$
$$\vec{v_2} = \frac{2m_1}{m_1 + m_2}\vec{u_1}$$

ii) Let the particles have same mass, then
$$\vec{v_1} = \vec{u_2}$$
$$\vec{v_2} = \vec{u_1}$$
Here, the final velocities just exchange their values with the initial velocities.

iii) Let the particle at rest be massive. ($\vec{u_2} = 0$ & $m_2 \gg m_1$). Hence,
$$\vec{v_1} = -\vec{u_1}$$
$$\vec{v_2} = 0$$
Here, the massive particle remains at rest and the lighter particle rebounds with the same velocity with which it collided.

iv) Let the particle at rest be light. ($\vec{u_2} = 0$ & $m_2 \ll m_1$). Hence,
$$\vec{v_1} = \vec{u_1}$$
$$\vec{v_2} = 2\vec{u_1}$$
Here, the massive particle moves with it initial velocity and the lighter particle moves with twice the initial velocity of the massive particle.

Centre of Mass Frame (One – Dimension)

The centre of mass for a two particle system is given as,
$$\vec{V_{cm}} = \frac{m_1\vec{u_1} + m_2\vec{u_2}}{m_1 + m_2}$$
According to Galilean transformation, velocity before collision relative to the centre of mass frame is given as,
$$\vec{u_1'} = \vec{u_1} - \vec{V_{cm}} = \vec{u_1} - \frac{m_1\vec{u_1} + m_2\vec{u_2}}{m_1 + m_2}$$
$$\Rightarrow \vec{u_1'} = \frac{m_2}{m_1 + m_2}(\vec{u_1} - \vec{u_2})$$

$$\overrightarrow{u_2'} = \overrightarrow{u_2} - \overrightarrow{V_{cm}} = \overrightarrow{u_2} - \frac{m_1\overrightarrow{u_1} + m_2\overrightarrow{u_2}}{m_1 + m_2}$$

$$\Rightarrow \overrightarrow{u_2'} = \frac{m_1}{m_1 + m_2}(\overrightarrow{u_2} - \overrightarrow{u_1})$$

After collision, the velocities are given as

$$\overrightarrow{v_1'} = \overrightarrow{v_1} - \overrightarrow{V_{cm}} = \frac{m_1 - m_2}{m_1 + m_2}\overrightarrow{u_1} + \frac{2m_2}{m_1 + m_2}\overrightarrow{u_2} - \frac{m_1\overrightarrow{u_1} + m_2\overrightarrow{u_2}}{m_1 + m_2}$$

$$\overrightarrow{v_1'} = -\frac{m_2}{m_1 + m_2}(\overrightarrow{u_1} - \overrightarrow{u_2})$$

$$\overrightarrow{v_2'} = \overrightarrow{v_2} - \overrightarrow{V_{cm}} = \frac{m_2 - m_1}{m_1 + m_2}\overrightarrow{u_2} + \frac{2m_1}{m_1 + m_2}\overrightarrow{u_1} - \frac{m_1\overrightarrow{u_1} + m_2\overrightarrow{u_2}}{m_1 + m_2}$$

$$\overrightarrow{v_1'} = -\frac{m_1}{m_1 + m_2}(\overrightarrow{u_2} - \overrightarrow{u_1})$$

Hence, we see that in the centre of mass frame, the initial and final velocities are same in magnitude with just change in direction.

Let one of the mass be initially at rest. Say $\overrightarrow{u_2} = 0$. Hence, the initial velocities in centre of mass frame are given as,

$$\overrightarrow{u_1'} = \frac{m_2}{m_1 + m_2}\overrightarrow{u_1}$$

$$\overrightarrow{u_2'} = -\frac{m_1}{m_1 + m_2}\overrightarrow{u_1}$$

In the centre of mass frame, the centre of mass is at rest. Hence, the linear momentum is conserved and is zero. Thus,

$$m_1\overrightarrow{u_1'} + m_2\overrightarrow{u_2'} = 0$$

$$m_1\overrightarrow{v_1'} + m_2\overrightarrow{v_2'} = 0$$

$$\Rightarrow \overrightarrow{u_1'} = -\frac{m_2}{m_1}\overrightarrow{u_2'}$$

$$\Rightarrow \overrightarrow{v_1'} = -\frac{m_2}{m_1}\overrightarrow{v_2'}$$

Since, the sum of kinetic energies is conserved, we get

$$\frac{1}{2}m_1\overrightarrow{u_1'^2} + \frac{1}{2}m_2\overrightarrow{u_2'^2} = \frac{1}{2}m_1\overrightarrow{v_1'^2} + \frac{1}{2}m_2\overrightarrow{v_2'^2}$$

$$\Rightarrow \frac{1}{2}m_1\left(\frac{m_2}{m_1}\overrightarrow{u_2'}\right)^2 + \frac{1}{2}m_2\overrightarrow{u_2'^2} = \frac{1}{2}m_1\left(\frac{m_2}{m_1}\overrightarrow{v_2'}\right)^2 + \frac{1}{2}m_2\overrightarrow{v_2'^2}$$

$$\overrightarrow{u_2'^2}\left[\frac{m_2^2}{m_1^2} + m_2\right] = \overrightarrow{v_2'^2}\left[\frac{m_2^2}{m_1^2} + m_2\right]$$

$$\overrightarrow{u_2'} = \overrightarrow{v_2'}$$

Also, we get

$$\Rightarrow \frac{1}{2}m_1\overrightarrow{u_1'^2} + \frac{1}{2}m_2\left(\frac{m_1}{m_2}\overrightarrow{u_1'}\right)^2 = \frac{1}{2}m_1\overrightarrow{v_1'^2} + \frac{1}{2}m_2\left(\frac{m_1}{m_2}\overrightarrow{v_1'}\right)^2$$

$$\overrightarrow{u_1'^2}\left[\frac{m_1^2}{m_2^2} + m_1\right] = \overrightarrow{v_1'^2}\left[\frac{m_1^2}{m_2^2} + m_1\right]$$

$$\overrightarrow{u_1'} = \overrightarrow{v_1'}$$

Hence, the velocities remain same in magnitude but there is a change in the direction with respect to the centre of mass.

Perfectly Inelastic Collision

Laboratory Frame (One – Dimension)

In a perfectly inelastic collision, the particles stick together after the collision. Hence by principle of conservation of linear momentum, we get

$$m_1\overrightarrow{u_1} + m_2\overrightarrow{u_2} = (m_1 + m_2)\vec{v}$$

$$\Rightarrow \vec{v} = \frac{m_1\overrightarrow{u_1} + m_2\overrightarrow{u_2}}{m_1 + m_2}$$

Let one of the particles (say m_2) be at rest. $\overrightarrow{u_2} = 0$.

$$\Rightarrow \vec{v} = \frac{m_1\overrightarrow{u_1}}{m_1 + m_2}$$

Let $\overrightarrow{K_1}$ and $\overrightarrow{K_2}$ be the initial and final kinetic energies respectively. They are given as,

$$\overrightarrow{K_1} = \frac{1}{2}m_1\overrightarrow{u_1^2}$$

$$\overrightarrow{K_2} = \frac{1}{2}(m_1 + m_2)\overrightarrow{v^2} = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1\overrightarrow{u_1}}{m_1 + m_2}\right)^2 = \frac{1}{2}\frac{m_1^2\overrightarrow{u_1^2}}{m_1 + m_2}$$

We find that,

$$\frac{\overrightarrow{K_2}}{\overrightarrow{K_1}} = \frac{m_1}{m_1 + m_2}$$

Thus, $\overrightarrow{K_2} < \overrightarrow{K_1}$ as $m_1 < m_1 + m_2$.

Hence, the loss in kinetic energy is given as,

$$\vec{E} = \overrightarrow{K_1} - \overrightarrow{K_2} = \frac{1}{2}m_1\overrightarrow{u_1^2} - \frac{1}{2}\frac{m_1^2\overrightarrow{u_1^2}}{m_1 + m_2}$$

$$\vec{E} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\overrightarrow{u_1^2} = \frac{1}{2}\mu\,\overrightarrow{u_1^2}$$

Centre of Mass Frame (One – Dimension)

The centre of mass for a two particle system is given as,

$$\overrightarrow{V_{cm}} = \frac{m_1\overrightarrow{u_1} + m_2\overrightarrow{u_2}}{m_1 + m_2}$$

As $\overrightarrow{u_2} = 0$, we get

$$\overrightarrow{V_{cm}} = \frac{m_1\overrightarrow{u_1}}{m_1 + m_2}$$

The initial velocities in the centre of mass frame are given as,

$$\overrightarrow{u_1'} = \overrightarrow{u_1} - \overrightarrow{V_{cm}} = \overrightarrow{u_1} - \frac{m_1\overrightarrow{u_1}}{m_1 + m_2}$$

$$\overrightarrow{u_1'} = \frac{m_2}{m_1 + m_2}\overrightarrow{u_1}$$

$$\overrightarrow{u_2'} = \overrightarrow{u_2} - \overrightarrow{V_{cm}} = \overrightarrow{u_2} - \frac{m_1\overrightarrow{u_1}}{m_1 + m_2}$$

$$\overrightarrow{u_2'} = -\frac{m_1}{m_1 + m_2}\overrightarrow{u_1}$$

In the centre of mass frame, the centre of mass is at rest. Hence, the linear momentum is conserved and is zero. Thus,

$$m_1\overrightarrow{u_1'} + m_2\overrightarrow{u_2'} = 0$$
$$m_1\overrightarrow{u_1'} = -m_2\overrightarrow{u_2'}$$

The final velocity in the centre of mass frame is given as,

$$\overrightarrow{v'} = \vec{v} - \overrightarrow{V_{cm}} = \frac{m_1\overrightarrow{u_1}}{m_1 + m_2} - \frac{m_1\overrightarrow{u_1}}{m_1 + m_2}$$
$$\overrightarrow{v'} = 0$$

Let $\overrightarrow{K_1'}$ and $\overrightarrow{K_2'}$ be the initial and final kinetic energies respectively. They are given as,

$$\overrightarrow{K_1'} = \frac{1}{2}m_1\overrightarrow{u_1'^{\,2}} + \frac{1}{2}m_2\overrightarrow{u_2'^{\,2}}$$
$$\overrightarrow{K_2'} = 0$$

Hence, the loss in kinetic energy is given as,

$$\vec{E} = \overrightarrow{K_1'} - \overrightarrow{K_2'} = \frac{1}{2}m_1\overrightarrow{u_1'^{\,2}} + \frac{1}{2}m_2\overrightarrow{u_2'^{\,2}}$$
$$\vec{E} = \frac{1}{2}m_1\left(\frac{m_2}{m_1 + m_2}\overrightarrow{u_1}\right)^2 + \frac{1}{2}m_2\left(-\frac{m_1}{m_1 + m_2}\overrightarrow{u_1}\right)^2$$
$$\vec{E} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\overrightarrow{u_1^2} = \frac{1}{2}\mu\,\overrightarrow{u_1^2}$$

Hence, we see that the loss of energy in both laboratory and centre of mass frame is same.

Conservation Laws

Linear Momentum

The linear momentum is defined as the product of its mass and linear velocity. i.e.,

$$\vec{p} = m\vec{v}$$

The total linear momentum of a system of particles free from the action of external forces and subjected only to their mutual interaction remains constant, no matter how complicated the forces are.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

In an isolated system, $\vec{F} = 0$. Hence the linear momentum remains conserved.

Let us consider a system of two particles which are acted upon by the gravitational force. Let $\overrightarrow{F_{12}}$ be the force of particle 1 on particle 2 and Let $\overrightarrow{F_{21}}$ be the force of particle 2 on particle 1. Hence,

$$\overrightarrow{F_{12}} = -\overrightarrow{F_{21}}$$

Now,

$$\overrightarrow{F_{21}} = \frac{d\overrightarrow{p_2}}{dt} = \frac{d}{dt}(m_2\overrightarrow{v_2})$$
$$\overrightarrow{F_{12}} = \frac{d\overrightarrow{p_1}}{dt} = \frac{d}{dt}(m_1\overrightarrow{v_1})$$

But,

$$\overrightarrow{F_{12}} + \overrightarrow{F_{21}} = 0$$
$$\frac{d}{dt}(\overrightarrow{p_1} + \overrightarrow{p_2}) = 0$$
$$\overrightarrow{p_1} + \overrightarrow{p_2} = constant$$

Thus, we see that linear momentum is conserved.

Angular Momentum

The angular momentum of a particle about a fixed point is defined as the moment of its linear momentum about that point.

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})$$

Thus, angular momentum is perpendicular to the plane containing the position vector $\vec{r}$ and the velocity $\vec{v}$. If $\vec{\omega}$ is the angular velocity then taking magnitudes, we find $v = r\omega$.

$$mvr = mr^2\omega$$

$$|\vec{J}| = mr^2\omega = mr^2\dot{\theta}$$

Let us consider a rigid body rotating with angular velocity ω about an axis. Let us consider two particles having mass m_1 and m_2 at a distance of r_1 and r_2 from the axis. The sum of the moments of the linear momentum of all the particles of a rotating rigid body about the axis of rotation is called the angular momentum. Hence, we get

$$v_1 = r_1\omega \ \& \ v_2 = r_2\omega$$

The linear momentum is given by

$$p_1 = mr_1\omega \ \& \ p_2 = mr_2\omega$$

The angular momentum is given by

$$J_1 = mr_1^2\omega \ \& \ J_2 = mr_2^2\omega$$

The total angular momentum for a system of particles is given by

$$J = mr_1^2\omega + mr_2^2\omega + \cdots = \sum mr^2\omega = I\omega$$

where, I is defined as the moment of inertia.

If a force $\vec{F}$ acts on a particle at a point P whose position with respect to the origin of the inertial frame is given by the displacement vector $\vec{r}$, then torque is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

We know that angular momentum is given by,

$$\vec{J} = \vec{r} \times \vec{p}$$

Differentiating with respect to time t, we get

$$\frac{d\vec{J}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

But, $\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times (m\vec{v}) = 0$

$$\Longrightarrow \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$$

$$\Longrightarrow \vec{\tau} = \frac{d\vec{J}}{dt}$$

Hence, torque is defined as the rate of change of angular momentum.

For a rigid body, $\vec{J} = I\vec{\omega}$. Hence,

$$\vec{\tau} = \frac{d(I\vec{\omega})}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$$

where, $\vec{\alpha}$ is termed as the angular acceleration.

If no torque acts, then

$$I\vec{\alpha} = 0$$

$$\Longrightarrow \vec{\alpha} = 0$$

$$\Longrightarrow \frac{d\vec{\omega}}{dt} = 0$$

$$\Longrightarrow \vec{\omega} = constant$$

Hence when no torque acts on a rigid body, angular velocity is conserved.

Angular Impulse

If an unbalanced external torque $\vec{\tau}$ is applied to a body rotating with an angular velocity $\overrightarrow{\omega_1}$ for a time t so that angular velocity changes to $\overrightarrow{\omega_2}$, then

$$\vec{\alpha} = \frac{\overrightarrow{\omega_2} - \overrightarrow{\omega_1}}{t}$$

The torque is given by

$$\vec{\tau} = I\vec{\alpha} = I\left(\frac{\overrightarrow{\omega_2} - \overrightarrow{\omega_1}}{t}\right)$$

$$t\,\vec{\tau} = I(\overrightarrow{\omega_2} - \overrightarrow{\omega_1}) = Id\omega$$

This quantity is called as the angular impulse.

Conservation of Energy

Let the force be defined as a function of potential energy U and position vector r. Hence,

$$\vec{F} = -\left(\frac{\partial U}{\partial r}\right)\hat{r}$$

Where, potential energy is a function of force F and position vector r. Hence,

$$U = U(F, r) \text{ and } r = r(r)$$

The kinetic energy $K = \frac{1}{2}mv^2$ also does not explicitly depend on time t. Hence, we get

$$\frac{\partial U}{\partial t} = 0 \;\&\; \frac{\partial K}{\partial t} = 0$$

The total energy $E = U + K$ is also defined as,

$$E = E(F, r)$$

Taking differential, we get

$$dE = \frac{\partial E}{\partial r}dr + \frac{\partial E}{\partial t}dt$$

$$= \frac{\partial (U+K)}{\partial r}dr + \frac{\partial (U+K)}{\partial t}dt$$

$$= \left(\frac{\partial U}{\partial r} + \frac{\partial K}{\partial r}\right)dr + \left(\frac{\partial U}{\partial t} + \frac{\partial K}{\partial t}\right)dt$$

$$dE = \left(\frac{\partial U}{\partial r} + \frac{\partial K}{\partial r}\right)dr$$

$$\frac{dE}{dt} = \left(\frac{\partial U}{\partial r} + \frac{\partial K}{\partial r}\right)\frac{dr}{dt}$$

But, $\frac{\partial U}{\partial r} = -F$ and $\frac{\partial K}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{2}mv^2\right) = mv\frac{\partial v}{\partial r}$

But as v is not a function of t, $\frac{\partial v}{\partial r} = \frac{dv}{dr}$.

Hence,

$$\frac{\partial K}{\partial r} = mv\frac{dv}{dr} = m\frac{dr}{dt}\frac{dv}{dr} = m\frac{dv}{dt} = ma$$

Therefore, we get

$$\frac{dE}{dt} = (-F + ma)\frac{dr}{dt}$$

But $F = ma$, hence,

$$\frac{dE}{dt} = 0 \quad \Longrightarrow \; E = constant$$

Thus, energy is conserved.

ROCKET

Rocket propulsion is based on the principle of conservation of momentum. When the rocket is fixed, the exhaust gases rush downward at a high speed and pushes the rocket forward.

Let us suppose, at any instant

Rocket's mass along with its fuel = m and velocity of rocket = v

After a time Δt, the mass ejected be Δm with an exhaust velocity $\overrightarrow{v_e}$ with respect to moving rocket. Velocity $\overrightarrow{v_e}$ is constant and an intrinsic negative quantity. Therefore, velocity of ejected gases with respect to earth is given as,

$$\overrightarrow{v_o} = \vec{v} + \overrightarrow{v_e}$$

As mass decreases to $m - \Delta m$, velocity increases to $v + \Delta v$.

Let,

Initial momentum, $\overrightarrow{p_1} = m\vec{v}$

Final momentum, $\overrightarrow{p_2} = (m - \Delta m)(\vec{v} + \Delta\vec{v}) + \Delta m(\vec{v} + \overrightarrow{v_e})$

Hence,

$$\Delta\vec{p} = \overrightarrow{p_2} - \overrightarrow{p_1} = m\Delta\vec{v} - \Delta m\Delta\vec{v} + \Delta m\overrightarrow{v_e}$$

But as Δm is a very small quantity, we can neglect the product $\Delta m\Delta\vec{v}$. Therefore, we get

$$\Delta\vec{p} = m\Delta\vec{v} + \Delta m\overrightarrow{v_e}$$

Taking limits, we get

$$\lim_{\Delta t\to 0}\frac{\Delta\vec{p}}{\Delta t} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + \overrightarrow{v_e}\frac{dm}{dt}$$

$\frac{dm}{dt}$ is an intrinsic negative quantity. Taking magnitudes, we get

$$\frac{dp}{dt} = m\frac{dv}{dt} - v_e\frac{dm}{dt}$$

Now, rate of change of momentum is defined as the external force applied on the rocket. Hence,

$$\frac{dp}{dt} = F_e$$

$$\Rightarrow F_e = m\frac{dv}{dt} - v_e\frac{dm}{dt}$$

Hence, the net force is given by,

$$m\frac{dv}{dt} = F_e + v_e\frac{dm}{dt}$$

If the rocket is outside the gravitational pull of the earth, $F_e = 0$. Hence, we get

$$m\frac{dv}{dt} = v_e\frac{dm}{dt}$$

$v_e\frac{dm}{dt}$ is the reaction force and is the thrust which pushes the rocket forward.

Since, v_e & $\frac{dm}{dt}$ are both negative quantities, $v_e\frac{dm}{dt}$ is positive and is directed upward.

Near the earth, the upward thrust on the rocket is opposed by the force due to gravity. Hence, $F_e = mg$.

$$\Rightarrow m\frac{dv}{dt} = mg + v_e\frac{dm}{dt}$$

Hence for producing a large thrust, exhaust velocity v_e should be large as should the rate of change of mass $\frac{dm}{dt}$.

To get the equation of motion of a rocket far away from earth, we solve the equation

$$d\vec{v} = \overrightarrow{v_e}\frac{dm}{m}$$

If m_0 is the initial mass and v_0 is the initial velocity, then

$$\int_{\vec{v_0}}^{\vec{v}} d\vec{v} = \vec{v_e} \int_{m_0}^{m} \frac{dm}{m}$$

$$\vec{v} - \vec{v_0} = \vec{v_e} \log_e \left(\frac{m}{m_0}\right)$$

$$\vec{v} = \vec{v_0} + \vec{v_e} \log_e \left(\frac{m}{m_0}\right)$$

Taking magnitudes, we get

$$v = v_0 - v_e \log_e \left(\frac{m}{m_0}\right)$$

$$v = v_0 + v_e \log_e \left(\frac{m_0}{m}\right)$$

This is the equation of motion of a rocket at any instant.
If m_f is the mass of the burnt out rocket and $\vec{v_f}$ be the final velocity, then

$$\vec{v_f} - \vec{v_0} = \vec{v_e} \log_e \left(\frac{m}{m_0}\right)$$

Let us take, $\vec{v_0} = 0$, then

$$\vec{v_f} = \vec{v_e} \log_e \left(\frac{m}{m_0}\right)$$

Velocity Achieved by a Rocket at Height

Let the initial velocity of the rocket be v_0, R be the radius of the earth, m be the mass of the rocket and M be the mass of earth.
At the earth's surface,
Kinetic energy of the rocket, $K_s = \frac{1}{2} m v_0^2$
Potential energy of the rocket, $P_s = -\frac{GmM}{R}$
When the rocket reaches height, h
Kinetic energy of the rocket, $K_h = \frac{1}{2} m v^2$
Potential energy of the rocket, $P_h = -\frac{GmM}{R+h}$
According to law of conservation of energy,

$$\frac{1}{2} m v_0^2 - \frac{GmM}{R} = \frac{1}{2} m v^2 - \frac{GmM}{R+h}$$

$$v_0^2 - v^2 = 2GM \left[\frac{1}{R} - \frac{1}{R+h}\right]$$

But, acceleration due gravity is given as, $g = \frac{GM}{R^2}$. Hence, we get

$$v_0^2 - v^2 = \frac{2gh}{1 + h/R}$$

This is the expression for velocity of rocket at a height h.

Bibliography:

1) Simplified Course in Mechanics and Properties of Matter – C. L. Arora
2) Physics For Degree Students – C. L. Arora, P. S. Hemne