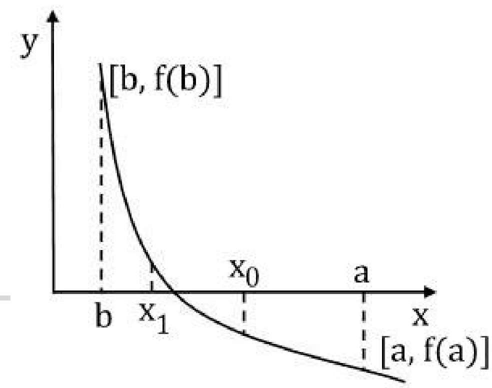
BISECTION METHOD

If a function $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root between a and b. It is shown graphically as,



Let $f(a)$ be negative and $f(b)$ be positive. We bisect the interval (a, b) and denote the mid – point as

$$x_0 = \frac{a+b}{2}$$

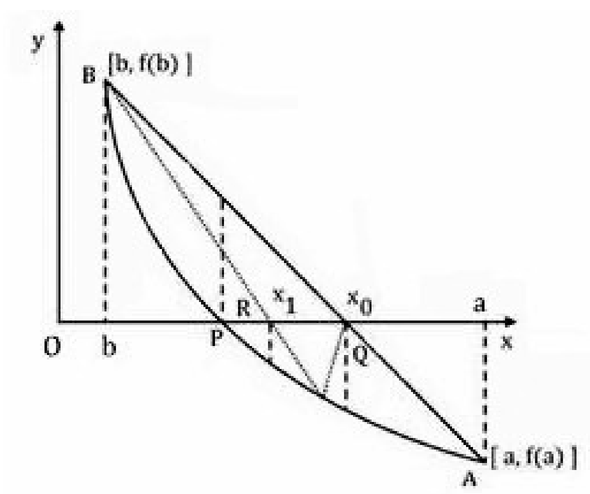
If $f(x_0) = 0$, then we have a root. Otherwise the root lies between either x_0 and b or between x_0 and a depending on whether $f(x_0)$ is negative or positive. We bisect this interval and find x_1.

If $f(x_1) = 0$, then we have a root. Otherwise, we repeatedly go on bisecting the intervals until we get a result which gives $f(x_n) = 0$ or $f(x_n) \approx 0$. The method guarantees that the iterative process will converge. By repeating this bisection procedure, we always enclose the root in a search interval and the interval is halved in each iteration. Thus ten steps will reduce the search interval by $2^{10} \approx 1000$.

FALSE POSITION METHOD (REGULA FALSI)

Bisection method even though effective is slow. False Position method or method of regula falsi is one such method which is faster and guarantees convergence.

Let $f(x) = 0$. Let the curve $y = f(x)$ be represented by AB with the condition that $f(a)$ at A and $f(b)$ at B are of opposite signs. Hence, a root must lie between these points. Let the curve cut X-axis at P. Then the root is OP. It is shown graphically as,



We now connect the two points A and B with a straight line. False position of curve AB is taken as the chord AB. Chord AB cuts X-axis at Q. Hence, the approximate root of $f(x) = 0$ is OQ. The extremities A and B are given as $[a, f(a)]$ and $[b, f(b)]$ respectively. The equation of the chord AB is given as,

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

The point of intersection is given by putting $y = 0$. Hence, we get,

$$-f(a) = \frac{f(b) - f(a)}{b - a}\,(x - a)$$

$$\Rightarrow x = a + \frac{(a - b)f(a)}{f(b) - f(a)}$$

$$\Rightarrow x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Let this value be x_0. If $f(x_0) = 0$, then we have a root. Otherwise the root lies between either x_0 and b or between x_0 and a depending on whether $f(x_0)$ is negative or positive. We again follow the same procedure as above and go on finding $x_n{}'$s until we get a result which gives $f(x_n) = 0$ or $f(x_n) \approx 0$. The method guarantees that the iterative process will converge.

NEWTON RAPHSON METHOD

This method is generally used to improve the result obtained by one of the other methods. Let x_0 be an approximate root of $f(x) = 0$. Let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$. Expanding $f(x_0 + h)$ by Taylor's series, we get

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots = 0$$

Neglecting the second and higher order derivatives, we have

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

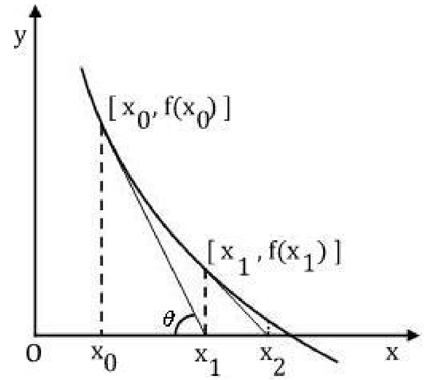
Hence, a better approximation x_1 is given by,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are given by $x_2, x_3, \ldots\ldots, x_n, x_{n+1}$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $f(x)$ be given by the curve as shown in the figure. We start with point x_0 and determine the slope $f'(x_0)$ at x_0.



The next approximation to the root x_1 is given as the point where the tangent cuts the X-axis. The equation is given as,

$$\frac{f(x_0)}{x_0 - x_1} = \tan\theta = f'(x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We again follow the same procedure as above and go on finding x_n's until we get a result which gives $f(x_n) = 0$ or $f(x_n) \approx 0$. The method is sensitive to starting approximation and converges faster if starting approximation is near the root.

SECANT METHOD

A serious disadvantage of the Newton – Raphson method is the need to calculate $f'(x)$ in each iteration. There are situations where a closed form for $f'(x)$ is not available. Thus methods which do not need the evaluation of $f'(x)$ are sought. One of these is the Secant method. In this method $f'(x)$ is approximated by the expression,

$$f'(x) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

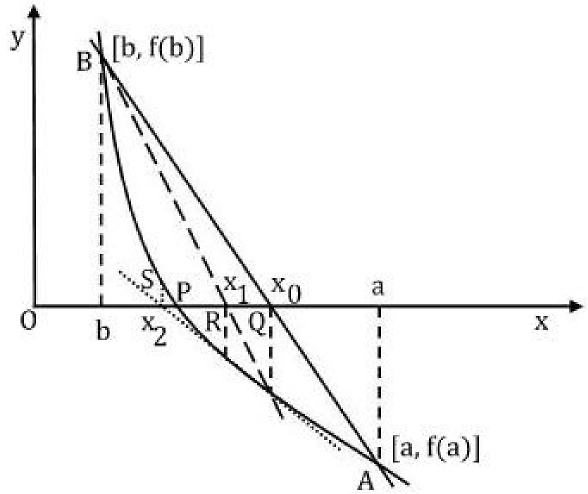
where x_n and x_{n-1} are the two approximations to the root. The iterative formula for $(n+1)$th approximation is given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{[f(x_n) - f(x_{n-1})]/[x_n - x_{n-1}]}$$

For computation, the formula is not suitable as it involves division by a small quantity and would lead to loss of significant digits. The above formula can be written as,

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Let $f(x) = 0$. Let the curve $y = f(x)$ be represented by AB having values $f(a)$ at A and $f(b)$ at B. Let the curve cut X-axis at P. Then the root is OP. It is shown graphically as,



We connect points A and B with a secant. The point where it cuts the X-axis is $Q\ (x_0)$. Another secant is drawn through B and $f(x_0)$. This secant cuts the X-axis at $R\ (x_1)$. We again follow the same procedure as above and go on finding $x_n{}'$s until we get a result which gives $f(x_n) = 0$ or $f(x_n) \approx 0$. The method does not guarantee that the iterative process will converge if the initial approximation is not near the root.

ITERATION METHOD (METHOD OF SUCCESSIVE APPROXIMATIONS)

If the equation $f(x) = 0$ whose roots are to be found is expressed as $x = g(x)$ then an iterative method can be developed. The iterative technique is to guess a starting value x_0. Substitute it in $g(x)$. Take $g(x_0)$ as the next guess till two successive values of x are close enough. The iterative formula is given by,

$$x_1 = g(x_0)$$

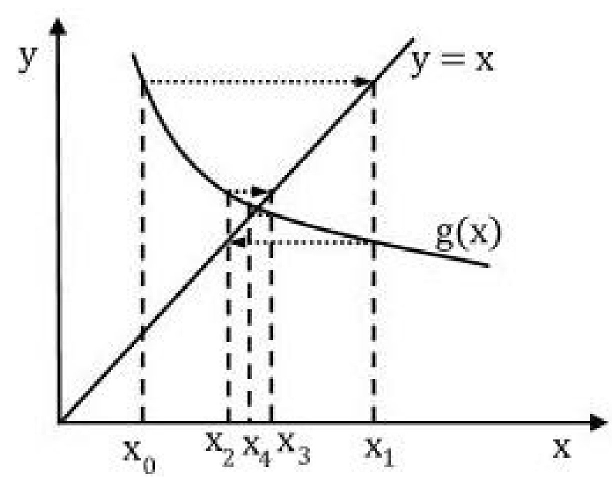
$$x_2 = g(x_1)$$

$$\vdots$$

$$x_{n+1} = g(x_n)$$

The method will converge only if $|g'(x_n)| < 1$.

The roots of the equation $f(x) = 0$ are the same as the points of intersection of the straight line $y = x$ and the function $g(x)$. This is as illustrated in the figure,



Theorem: Let $x = \alpha$ be the root of $f(x) = 0$ and let I be an interval containing the point $x = \alpha$. Let $g(x)$ and $g^{'}(x)$ be continuous in I where $g(x)$ is defined by the equation $x = g(x)$ which is equivalent to $f(x) = 0$. Then if $|g^{'}(x)| < 1$ for all x in I, the sequence of approximations x_0, x_1, x_2,..., x_n defined as $x_{n+1} = g(x_n)$ converges to the root α, provided that the initial approximation x_0 is chosen in I.

Proof: Since, α is a root of the equation $x = g(x)$, we have

$$\alpha = g(\alpha)$$

Since, $x_{n+1} = g(x_n)$, we get

$$x_1 = g(x_0)$$

On subtracting, we get

$$x_1 - \alpha = g(x_0) - g(\alpha)$$

Mean value theorem states that if $g(x)$ is continuous in $[a, b]$ and $g^{'}(x)$ exists in $[a, b]$, then there exists at least one value of x, say α, between a and b such that

$$g^{'}(\alpha) = \frac{g(b) - g(a)}{b - a}, \qquad a < \alpha < b$$

Using the Mean value theorem, we get

$$x_1 - \alpha = (x_0 - \alpha)g^{'}(\alpha_0), \qquad x_0 < \alpha_0 < \alpha$$

Similarly, we obtain

$$x_2 - \alpha = (x_1 - \alpha)g^{'}(\alpha_1), \qquad x_1 < \alpha_1 < \alpha$$

$$x_3 - \alpha = (x_2 - \alpha)g^{'}(\alpha_2), \qquad x_2 < \alpha_2 < \alpha$$

$$\vdots \qquad\qquad \vdots$$

$$x_{n+1} - \alpha = (x_n - \alpha)g^{'}(\alpha_n), \qquad x_n < \alpha_n < \alpha$$

Let us have $|g^{'}(x_i)| \leq k < 1$, for all i.

Multiplying the above equations, we get

$$x_{n+1} - \alpha = (x_0 - \alpha)g^{'}(\alpha_0)g^{'}(\alpha_1)g^{'}(\alpha_2)\dots g^{'}(\alpha_n)$$

Since, $|g^{'}(\alpha_i)| \leq k$, the above equation becomes,

$$|x_{n+1} - \alpha| \leq k^{n+1}|x_0 - \alpha|$$

As $n \to \infty$, the right-hand side tends to zero and it follows that the sequence of approximations x_0, x_1, x_2,..., x_n defined as $x_{n+1} = g(\alpha_n)$ converges to the root α if $k < 1$. If $k > 1$, the sequence will never converge to the root.

SOLUTION OF SYSTEMS OF NONLINEAR EQUATIONS

We can solve simultaneous nonlinear equations using the above methods. We will use Iteration and Newton – Raphson methods to solve the system of equations. For simplicity, we will consider the case of two equations in two unknowns.

ITERATION METHOD

Let the equation be given by,

$$f(x,y)=0 \text{ and } g(x,y)=0$$

whose real roots are required.

We assume that the equations may be written as,

$$x=F(x,y) \text{ and } y=G(x,y)$$

Where the functions F and G satisfy the following conditions in the neighbourhood of the root,

$$\left|\frac{\partial F}{\partial x}\right|+\left|\frac{\partial F}{\partial y}\right|<1 \text{ and } \left|\frac{\partial G}{\partial x}\right|+\left|\frac{\partial G}{\partial y}\right|<1$$

Let (x_0,y_0) be the initial approximation to a root (α,β). We then construct the successive approximations according to the following formulae,

$$\begin{aligned} x_1 &= F(x_0,y_0), & y_1 &= G(x_0,y_0) \\ x_2 &= F(x_1,y_1), & y_2 &= G(x_1,y_1) \\ x_3 &= F(x_2,y_2), & y_3 &= G(x_2,y_2) \\ &\vdots & &\vdots \\ x_{n+1} &= F(x_n,y_n), & y_{n+1} &= G(x_n,y_n) \end{aligned}$$

If the iteration process converges, then we obtain the roots in the given limit as,

$$\alpha=F(\alpha,\beta) \text{ and } \beta=G(\alpha,\beta)$$

NEWTON – RAPHSON METHOD

Let the equation be given by,

$$f(x,y)=0 \text{ and } g(x,y)=0$$

whose real roots are required.

Let (x_0,y_0) be the initial approximation to a root (α,β). If (x_0+h,y_0+k) is the root of the system, then we have

$$f(x_0+h,y_0+k)=0$$

$$g(x_0+h,y_0+k)=0$$

Assuming that f and g are sufficiently differentiable, we expand by Taylor's series to get

$$f(x_0,y_0)+h\frac{\partial f}{\partial x_0}+k\frac{\partial f}{\partial y_0}+\ \dots=0$$

$$g(x_0,y_0)+h\frac{\partial g}{\partial x_0}+k\frac{\partial g}{\partial y_0}+\ \dots=0$$

where, $\frac{\partial f}{\partial x_0} = \left[\frac{\partial f}{\partial x}\right]_{x=x_0}$ and $\frac{\partial f}{\partial y_0} = \left[\frac{\partial f}{\partial y}\right]_{y=y_0}$

Neglecting the second and higher order terms, we obtain the system of linear equations,

$$h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = -f(x_0, y_0)$$

$$h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = -g(x_0, y_0) \qquad \text{(A)}$$

If the Jacobian,

$$J(f,g) = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

does not vanish then the system of linear equations (A) possess a unique solution given by,

$$h = \begin{vmatrix} -f & \frac{\partial f}{\partial y} \\ -g & \frac{\partial g}{\partial y} \end{vmatrix} \Big/ J(f,g)$$

$$k = \begin{vmatrix} \frac{\partial f}{\partial x} & -f \\ \frac{\partial g}{\partial x} & -g \end{vmatrix} \Big/ J(f,g)$$

The new approximations are then given by,

$$x_1 = x_0 + h \text{ and } y_1 = y_0 + h$$

The process is repeated till we obtain the roots to the desired accuracy.

RATE OF CONVERGENCE

Let x_0, x_1, x_2,..., x_n,... be the successive approximations of an iteration process to find the root α of function $f(x)$. This sequence converges to root α with order $p \geq 1$ if,

$$|x_{n+1} - \alpha| \leq \lambda |x_n - \alpha|^p$$

p is called the order of convergence and λ is called the asymptotic error constant. If for each i, we denote the error by $\varepsilon = x_i - \alpha$. Then the above inequality be written as,

$$|\varepsilon_{i+1}| \leq \lambda |\varepsilon_i|^p$$

$$\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} \leq \lambda$$

This inequality shows the relationship between the errors in successive approximations. Suppose $p = 2$ and $|\varepsilon_i| = 10^{-2}$ for some i, then we can expect that $|\varepsilon_{i+1}| \leq \lambda(10^{-4})$. Thus if p is large, the iteration converges rapidly. When p takes the integer values 1, 2,3 then we say that the convergence is linear, quadratic and cubic respectively. Let us find the rate of convergence for the above mentioned methods.

BISECTION METHOD

Let the function be $f(x) = 0$ and we start with the interval $[a_0, b_0]$ which contains the root α. As we go on bisecting, the intervals get halved in every iteration. The root α is contained in each $[a_i, b_i]$ for $i = 0,1,2,\dots,n,\dots$

For any i, let $x_i = \frac{a_i+b_i}{2}$ denote the middle point of interval $[a_i, b_i]$. Then x_0, x_1, x_2,..., x_n,... are taken as successive approximations to the root α. Therefore, we have

$$|x_{i+1} - \alpha| \leq \frac{|x_i - \alpha|}{2}$$

$$|\varepsilon_{i+1}| \leq \frac{|\varepsilon_i|}{2}$$

$$\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|} \leq \frac{1}{2}$$

Comparing with $\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} \leq \lambda$, we get $p = 1$. Thus Bisection method converges linearly.

NEWTON – RAPHSON METHOD

The Newton – Raphson iteration for a function $f(x) = 0$ is given as,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Let α be the root of the equation and the error be denoted by ε_i in the ith iteration. Hence the above equation can be written as,

$$\alpha + \varepsilon_{i+1} = (\alpha + \varepsilon_i) - \frac{f(\alpha + \varepsilon_i)}{f'(\alpha + \varepsilon_i)}$$

By Taylor's series, we can expand $f(\alpha + \varepsilon_i)$ and $f'(\alpha + \varepsilon_i)$ around α. We have,

$$f(\alpha + \varepsilon_i) = f(\alpha) + \varepsilon_i f'(\alpha) + \frac{\varepsilon_i^2}{2!} f''(\alpha) + \cdots$$

$$f'(\alpha + \varepsilon_i) = f'(\alpha) + \varepsilon_i f''(\alpha) + \frac{\varepsilon_i^2}{2!} f'''(\alpha) + \cdots$$

As α is the root of the equation $f(x) = 0$, we have $f(\alpha) = 0$. Hence, we get

$$\varepsilon_{i+1} = \varepsilon_i - \frac{\varepsilon_i f'(\alpha) + \frac{\varepsilon_i^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + \varepsilon_i f''(\alpha) + \frac{\varepsilon_i^2}{2!} f'''(\alpha) + \cdots}$$

On dividing the numerator and denominator by $f'(\alpha)$ with the assumption that $f'(\alpha) \neq 0$, we obtain

$$\varepsilon_{i+1} = \varepsilon_i - \frac{\varepsilon_i + \frac{\varepsilon_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)} + \cdots}{1 + \varepsilon_i \frac{f''(\alpha)}{f'(\alpha)} + \frac{\varepsilon_i^2}{2} \frac{f'''(\alpha)}{f'(\alpha)} + \cdots}$$

If $f''(\alpha)/f'(\alpha)$ is finite and ε_i is small then higher powers of ε_i may be neglected. Hence, we get

$$\varepsilon_{i+1} = \varepsilon_i - \frac{\varepsilon_i\left(1 + \frac{\varepsilon_i}{2}\frac{f''(\alpha)}{f'(\alpha)}\right)}{1 + \varepsilon_i\frac{f''(\alpha)}{f'(\alpha)}}$$

$$\varepsilon_{i+1} = \varepsilon_i - \varepsilon_i\left(1 + \frac{\varepsilon_i}{2}\frac{f''(\alpha)}{f'(\alpha)}\right)\left(1 - \varepsilon_i\frac{f''(\alpha)}{f'(\alpha)}\right)$$

Higher powers of ε_i may be neglected. Hence, we get

$$\varepsilon_{i+1} = \frac{\varepsilon_i^2}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$$\frac{\varepsilon_{i+1}}{\varepsilon_i^2} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

Thus, if $[f''(\alpha)/f'(\alpha)]$ is finite and not zero then the Newton – Raphson method converges to the root. Comparing with $\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} \le \lambda$, we get $p = 2$. Hence, the Newton – Raphson method is second order convergent. Physically second order convergence means that in each iteration, as we approach the root, the number of significant digits in the approximation doubles.

SECANT METHOD

The Secant method iteration for a function $f(x) = 0$ is given as,

$$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Let α be the root of the equation and the error be denoted by ε_i in the ith iteration. Hence the above equation can be written as,

$$\alpha + \varepsilon_{i+1} = \frac{(\alpha + \varepsilon_{i-1})f(\alpha + \varepsilon_i) - (\alpha + \varepsilon_i)f(\alpha + \varepsilon_{i-1})}{f(\alpha + \varepsilon_i) - f(\alpha + \varepsilon_{i-1})}$$

$$\alpha + \varepsilon_{i+1} = \frac{\alpha[f(\alpha + \varepsilon_i) - f(\alpha + \varepsilon_{i-1})]}{f(\alpha + \varepsilon_i) - f(\alpha + \varepsilon_{i-1})} + \frac{(\varepsilon_{i-1})f(\alpha + \varepsilon_i) - (\varepsilon_i)f(\alpha + \varepsilon_{i-1})}{f(\alpha + \varepsilon_i) - f(\alpha + \varepsilon_{i-1})}$$

$$\varepsilon_{i+1} = \frac{(\varepsilon_{i-1})f(\alpha + \varepsilon_i) - (\varepsilon_i)f(\alpha + \varepsilon_{i-1})}{f(\alpha + \varepsilon_i) - f(\alpha + \varepsilon_{i-1})}$$

By Taylor's series, we can expand $f(\alpha + \varepsilon_i)$ and $f(\alpha + \varepsilon_{i-1})$ around α. We have,

$$f(\alpha + \varepsilon_i) = f(\alpha) + \varepsilon_i f'(\alpha) + \frac{\varepsilon_i^2}{2!}f''(\alpha) + \cdots$$

$$f(\alpha + \varepsilon_{i-1}) = f(\alpha) + \varepsilon_{i-1} f'(\alpha) + \frac{\varepsilon_{i-1}^2}{2!}f''(\alpha) + \cdots$$

As α is the root of the equation $f(x) = 0$, we have $f(\alpha) = 0$. Hence, we get

$$\varepsilon_{i+1} = \frac{(\varepsilon_{i-1})\left[\varepsilon_i f'(\alpha) + \frac{\varepsilon_i^2}{2} f''(\alpha)\right] - (\varepsilon_i)\left[\varepsilon_{i-1} f'(\alpha) + \frac{\varepsilon_{i-1}^2}{2} f''(\alpha)\right]}{\left[\varepsilon_i f'(\alpha) + \frac{\varepsilon_i^2}{2} f''(\alpha)\right] - \left[\varepsilon_{i-1} f'(\alpha) + \frac{\varepsilon_{i-1}^2}{2} f''(\alpha)\right]}$$

$$\varepsilon_{i+1} = \frac{(\varepsilon_{i-1})\left[\frac{\varepsilon_i^2}{2} f''(\alpha)\right] - (\varepsilon_i)\left[\frac{\varepsilon_{i-1}^2}{2} f''(\alpha)\right]}{\left[\varepsilon_i f'(\alpha) + \frac{\varepsilon_i^2}{2} f''(\alpha)\right] - \left[\varepsilon_{i-1} f'(\alpha) + \frac{\varepsilon_{i-1}^2}{2} f''(\alpha)\right]}$$

$$\varepsilon_{i+1} = \frac{\frac{(\varepsilon_i \varepsilon_{i-1})}{2}(\varepsilon_i - \varepsilon_{i-1}) f''(\alpha)}{\left[\varepsilon_i f'(\alpha) + \frac{\varepsilon_i^2}{2} f''(\alpha)\right] - \left[\varepsilon_{i-1} f'(\alpha) + \frac{\varepsilon_{i-1}^2}{2} f''(\alpha)\right]}$$

Neglecting higher powers of ε_i, we get

$$\varepsilon_{i+1} = \frac{\frac{(\varepsilon_i \varepsilon_{i-1})}{2}(\varepsilon_i - \varepsilon_{i-1}) f''(\alpha)}{(\varepsilon_i - \varepsilon_{i-1}) f'(\alpha)}$$

$$\varepsilon_{i+1} = \frac{(\varepsilon_i \varepsilon_{i-1})}{2} \frac{f''(\alpha)}{f'(\alpha)}$$

In order to find the order of convergence, it is necessary to find a formula of the type,

$$\varepsilon_{i+1} = \lambda \varepsilon_i^p$$

If,

$$\varepsilon_{i+1} = \lambda \varepsilon_i^p$$

Then,

$$\varepsilon_i = \lambda \varepsilon_{i-1}^p$$

Or,

$$\varepsilon_{i-1} = \frac{\varepsilon_i^{1/p}}{\lambda^{1/p}}$$

Hence, we get

$$\varepsilon_{i+1} = \left(\varepsilon_i \varepsilon_i^{\frac{1}{p}}\right) \frac{1}{2\lambda^{1/p}} \frac{f''(\alpha)}{f'(\alpha)}$$

Let the asymptotic error constant λ' be,

$$\lambda' = \frac{1}{2\lambda^{1/p}} \frac{f''(\alpha)}{f'(\alpha)}$$

Hence, we get

$$\varepsilon_{i+1} = \varepsilon_i^{\frac{p+1}{p}} \lambda'$$

As we know, $\varepsilon_{i+1} = \lambda \varepsilon_i^p$, we get

$$\lambda \varepsilon_i^p = \varepsilon_i^{\frac{p+1}{p}} \lambda'$$

Equating the powers of ε_i on both sides, we get

$$p = \frac{p+1}{p}$$

$$\Rightarrow p^2 - p - 1 = 0$$

The solution to this quadratic equation is,

$$p = \frac{1 \pm \sqrt{5}}{2}$$

Since p is a positive quantity, we get

$$p = \frac{1 + \sqrt{5}}{2} = 1.618$$

Hence,

$$\varepsilon_{i+1} = \varepsilon_i^{1.618} \left[\frac{1}{2\lambda^{\frac{1}{p}}} \frac{f''(\alpha)}{f'(\alpha)} \right]$$

Thus, if $[f''(\alpha)/f'(\alpha)]$ is small then the Secant method converges to the root. Comparing with $\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} \leq \lambda$, we get $p = 1.618$. Hence, the order of convergence of Secant method is 1.618. This method converges at a slower rate than Newton – Raphson method. However, the function is evaluated only once in each iteration as compared to two evaluations ($f(x)$ and $f'(x)$) in Newton – Raphson method. On the average it is found that Secant method is more efficient as compared to Newton – Raphson method.

FALSE POSITION METHOD

The derivation for the rate of convergence of False position method is same as that of Secant method and we obtain the relation,

$$\varepsilon_{i+1} = \frac{(\varepsilon_i \varepsilon_{i-1})}{2} \frac{f''(\alpha)}{f'(\alpha)}$$

In the case of False position method one of the points x_0 or x_1 of the interval (x_0, x_1), which contains the root α, is always fixed and the other point varies with i. If the point x_0 is fixed then the function $f(x)$ is approximated by the straight line passing through the points (x_0, f_0) and $(x_i, f_i), i = 1,2, \dots$. The error equation becomes

$$\varepsilon_{i+1} = \varepsilon_i \varepsilon_0 \frac{f''(\alpha)}{2\, f'(\alpha)}$$

where, $\varepsilon_0 = x_0 - \alpha$ is independent of i. Hence, we can write

$$\varepsilon_{i+1} = \varepsilon_i \lambda'$$

where, $\lambda' = \varepsilon_0 \frac{f''(\alpha)}{2\, f'(\alpha)}$

Comparing with $\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} \leq \lambda$, we get $p = 1$. Thus False position method converges linearly

ITERATION METHOD

Iteration method for $f(x) = 0$ is given as,

$$x_{i+1} = g(x_i)$$

with the condition,

$$|x_{i+1} - \alpha| \leq |g'(\alpha_i)||x_i - \alpha| \qquad x_0 < \alpha_i < \alpha$$

$$|\varepsilon_{i+1}| \leq |g'(\alpha_i)||\varepsilon_i|$$

As the approximations x_i approach the root α with each iteration i, $g'(\alpha_i)$ approaches a constant value $g'(\alpha)$. Hence, we get,

$$|\varepsilon_{i+1}| \leq |g'(\alpha)||\varepsilon_i|$$

$$\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|} = |g'(\alpha)|$$

Thus, if $|g'(\alpha)|$ is not zero then the Iteration method converges to the root. Comparing with $\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} \leq \lambda$, we get $p = 1$. Hence, the Iteration method converges linearly.

COMPARISON OF ITERATIVE METHODS

	Method	Iterative Formula	Order of Convergence	Reliability of Convergence
1.	Bisection	$x_{i+1} = \frac{x_i + x_{i-1}}{2}$ x_i and x_{i-1} enclose root.	1	Guaranteed Convergence
2.	False Position	$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$ x_i and x_{i-1} enclose root.	1	Guaranteed Convergence
3.	Newton – Raphson	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$	2	Sensitive to starting value. Convergence fast if starting point near the root.
4.	Secant	$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$ x_i and x_{i-1} need not enclose root.	1.618	No guarantee of convergence if starting value not near root. Method is economical.
5.	Iteration	$x_{i+1} = g(x_i)$	1	No guarantee of convergence. Easy to program.

GAUSSIAN ELIMINATION METHOD

This is an elementary elimination method and it reduces the system of linear equations to an equivalent upper triangular system which is solved by back substitution.

Let us consider a system of three equations in three unknowns,

$$a_{11}x + a_{12}y + a_{13}z = a_{14}$$

$$a_{21}x + a_{22}y + a_{23}z = a_{24}$$

$$a_{31}x + a_{32}y + a_{33}z = a_{34}$$

where x, y, z are the unknowns and a_{ij} $i = 1,2,3;$ $j = 1,2,3,4$ are the coefficients. We first form an augmented matrix of the coefficients of the equations. We get,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Now we need to eliminate the first element (a_{21} and a_{31}) from second and third equation. For eliminating a_{21}, we multiply the first equation by ${a_{21}}/{a_{11}}$ and subtract the equation from second equation. Similarly for eliminating a_{31}, we multiply the first equation by ${a_{31}}/{a_{11}}$ and subtract the equation from third equation. Thus we have,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{'} & a_{23}^{'} & a_{24}^{'} \\ 0 & a_{32}^{'} & a_{33}^{'} & a_{34}^{'} \end{bmatrix}$$

Now we need to eliminate the second element $a_{32}^{'}$ from third equation. For eliminating $a_{32}^{'}$, we multiply the second equation by ${a_{32}^{'}}/{a_{22}^{'}}$ and subtract the equation from third equation. Thus we have,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{'} & a_{23}^{'} & a_{24}^{'} \\ 0 & 0 & a_{33}^{''} & a_{34}^{''} \end{bmatrix}$$

The equations now can be represented as,

$$a_{11}x + a_{12}y + a_{13}z = a_{14}$$

$$a_{22}^{'}y + a_{23}^{'}z = a_{24}^{'}$$

$$a_{33}^{''}z = a_{34}^{''}$$

By back substitution, we can find the values of the unknowns x, y and z.

${a_{21}}/{a_{11}}$ and ${a_{31}}/{a_{11}}$ are called the multipliers of the first stage of elimination. In this stage, it is assumed that $a_{11} \neq 0$. The first equation is called as the pivotal equation and a_{11} is called as the first pivot. Similarly, ${a_{32}^{'}}/{a_{22}^{'}}$ is the multiplier for the second stage and $a_{22}^{'}$ is the second pivot.

The pivot element should not be zero or a small number. For maximum precision, the pivot element should be the largest in absolute value of all the elements in its column. i.e. $a_{11} > a_{22} > a_{33}$. Hence, during the Gauss elimination procedure, it should be taken into consideration and if it is not so, the equations should be interchanged so that the order as described above is maintained.

GAUSS –JORDAN ELIMINATION METHOD

A modification can be applied to Gaussian elimination method to obtain values of the unknowns directly. In the second stage, we can eliminate the second element a_{12} as well along with $a_{32}^{'}$. For eliminating a_{12}, we multiply the second equation by $a_{12}/a_{22}^{'}$ and subtract the equation from first equation. Thus we have,

$$\begin{bmatrix} a_{11}^{'} & 0 & a_{13}^{'} & a_{14}^{'} \\ 0 & a_{22}^{'} & a_{23}^{'} & a_{24}^{'} \\ 0 & 0 & a_{33}^{''} & a_{34}^{''} \end{bmatrix}$$

The equations now can be represented as,

$$a_{11}^{'}x \qquad + a_{13}^{'}z = a_{14}^{'}$$

$$a_{22}^{'}y + a_{23}^{'}z = a_{24}^{'}$$

$$a_{33}^{''}z = a_{34}^{''}$$

We can now find the values of the unknowns x, y and z.

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DIY (Do It Yourself)

1. Obtain a root, correct to three decimal places, using Bisection Method
   a. $x^3 + x^2 + x + 7 = 0$
   b. $e^x - 2 - x = 0$
   c. $2x - 3sinx - 5 = 0$
   d. $x^3 - x - 4 = 0$
   e. $xsinx + cosx = 0$
   f. $x^2 - 4sinx = 0$

2. Obtain a root, correct to three decimal places, using False Position Method
   a. $x^3 - 2x^2 + 3x - 5 = 0$
   b. $x^3 + 7x^2 + 9 = 0$
   c. $x\ sinx - 1 = 0$
   d. $x^3 + x - 1 = 0$
   e. $x^3 - 4x - 9 = 0$
   f. $x^3 - x^2 - x - 1 = 0$

3. Obtain a root, correct to three decimal places, using Newton - Raphson Method
   a. $x^3 - 5x + 3 = 0$
   b. $x^4 + x^2 - 80 = 0$
   c. $4(x - sinx) = 1$
   d. $x^3 - 4x + 1 = 0$
   e. $tanx + tanhx = 0$
   f. $value\ of\ \sqrt{2}$

4. Obtain a root, correct to three decimal places, using Secant Method
   a. $x^2 - 2x + 1 = 0$
   b. $x^3 + x^2 - 3x - 3 = 0$
   c. $cosx - xe^x = 0$
   d. $x + logx = 2$
   e. $x^3 - 4x + 1 = 0$
   f. $tanx - x = 0$

5. Obtain a root, correct to three decimal places, using Iteration Method
   a. $cosx = 3x - 1$
   b. $e^x tanx = 1$
   c. $e^{-x} = 10x$
   d. $e^x = x^2$
   e. $1 + logx = \frac{x}{2}$
   f. $sinx = \frac{x+1}{x-1}$

6. Obtain a root for the following systems of equations, correct to three decimal places
   a. $x^2 + y = 11$, $y^2 + x = 7$
   b. $x^3 = y + 100$, $y^3 = x + 150$
   c. $x^2 = 3xy - 7$, $y = 2(x + 1)$
   d. $x^2 + y^2 = 4$, $xy = 1$

7. Obtain a solution for the following systems of equations using Gaussian elimination method
   a. $3x + 2y + 4z = 7$, $2x + y + z = 7$, $x + 3y + 5z = 2$
   b. $5x - 2y + z = 4$, $7x + y - 5z = 8$, $3x + 7y + 4z = 10$
   c. $2x + 2y + z + 2u = 7$, $x - 2y - u = 2$, $3x - y - 2z - u = 3$, $x - 2u = 0$
   d. $2x + 3y - z = 5$, $4x + 4y - 3z = 3$, $-2x + 3y - z = 1$
   e. $x + y + z = 3$, $2x + 3y + z = 6$, $x - y - z = -3$
   f. $2x + 6y - z = -14$, $5x - y + 2z = 29$, $-3x - 4y + z = 4$